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A Method to Establish Non-Informative Prior Probabilities for Risk-Based Decision Analysis

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A Method to Establish Non-Informative Prior Probabilities for Risk-Based Decision Analysis

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Dedication

To my wife, Jihee

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A Method to Establish Non-Informative Prior Probabilities for Risk-Based Decision Analysis

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In Bayesian decision analysis, uncertainty and risk are accounted for with probabilities for the possible states, or states of nature, that affect the outcome of a decision. Application of Bayes' theorem requires non-informative prior probabilities, which represent the probabilities of states of nature for a decision maker under complete ignorance. These prior probabilities are then subsequently updated with any and all available information in assessing probabilities for making decisions. The conventional approach for the non-informative probability distribution is based on Bernoulli's principle of insufficient reason. This principle assigns a uniform distribution to uncertain states when a decision maker has no information about the states of nature. The principle of insufficient reason has three difficulties: it may inadvertently provide a biased starting point for decision making, it does not provide a consistent set of probabilities, and it violates reasonable axioms of decision theory.

The first objective of this study is to propose and describe a new method to establish non-informative prior probabilities for decision making under uncertainty.

The proposed decision-based method is focuses on decision outcomes that include preference in decision alternatives and decision consequences.

The second objective is to evaluate the logic and rationality basis of the proposed decision-based method. The decision-based method overcomes the three weaknesses associated with the principle of insufficient reason, and provides an unbiased starting point for decision making. It also produces consistent non-informative probabilities. Finally, the decision-based method satisfies axioms of decision theory that characterize the case of no information (or complete ignorance).

The third and final objective is to demonstrate the application of the decision-based method to practical decision making problems in engineering. Four major practical implications are illustrated and discussed with these examples. First, the method is practical because it is feasible in decisions with a large number of decision alternatives and states of nature and it is applicable to both continuous and discrete random variables of finite and infinite ranges. Second, the method provides an objective way to establish non-informative prior probabilities that capture a highly non-linear relationship between states of nature. Third, we can include any available information through Bayes' theorem by updating the non-informative probabilities without the need to assume more than is actually contained in the information. Lastly, two different decision making problems with the same states of nature may have different non-informative probabilities.

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Chapter 1. Introduction

1.1 BACKGROUND

Decision making is a fundamental aspect of engineering. When selecting the most preferred decision alternative, a decision maker builds a set of decision alternatives, seeks how much satisfaction or dissatisfaction would occur, and analyzes what is known and what is uncertain happen. Decision theory provides a rational framework to make decisions.

Because decisions are usually made under uncertainty, a probabilistic approach is used in decision theory to represent uncertainty. Bayesian decision theory is widely used in various fields of research and practice, including economics, business, medicine, and engineering. In a Bayesian decision framework, probabilities on the available information are established through Bayes' theorem. Bayes' theorem provides a logical method to combine a probability distribution with previous knowledge and new information.

Since the early age of probability theory, there has been controversy concerning the probability distribution with previous knowledge, which is represented by a prior probability distribution. The controversy is about the prior probability distribution under no previous knowledge (no information or complete ignorance), which is called a non-informative prior probability distribution. The issue has been pointed as a weakness of Bayesian approach because there has been no clear method to establish non-informative prior probabilities, yet these prior probabilities affect the updated or posterior probabilities and therefore the decision.

A conventional method for establishing non-informative prior probabilities is the principle of insufficient reason. The principle was used implicitly in Bayes (1763) and

Laplace (1814) and explicitly in Tribus (1969). There are extensions of the principle of insufficient reason: Jeffreys' rule (Jeffreys, 1961), and a data translated likelihood (Box and Tiao, 1973). Tribus (1969) indicated that the principle of insufficient reason is a special case of the application of the principle of maximum entropy, and Jaynes (1957), who proposed the principle of maximum entropy, addressed that the principle of maximum entropy may be regarded as an extension of the principle of insufficient reason.

However, the principle of insufficient reason has difficulties in the application to decision making problems. First, the principle of insufficient reason fails to provide a rational starting point for decision making. A non-informative prior probability distribution based on the principle of insufficient reason may yield a bias in a decision because it focuses on equally probable outcomes in uncertain variables. From this notion, decision alternatives may have different preferences over one another even in the state of complete ignorance. Second, non-informative prior probability distributions from the principle of insufficient reason may vary with the definition of a sample space for uncertain variables. The principle of insufficient reason assigns uniform probabilities to a sample space but using a different sample spaces may produce different probabilities. Efforts to define a unique sample space where the uniform probabilities should be assigned were made by Jeffreys (1961), Box and Tiao (1973), and Sinn (1980). However, there is no widely accepted and clear answer for the unique sample space. Third, the principle of insufficient reason fails to satisfy reasonable axioms for decision theory in the case of complete ignorance, as shown in Luce and Raiffa (1957). This study will propose a new method to establish non-informative prior probabilities for decision making under uncertainty, in an attempt to overcome these difficulties.

1.2 MOTIVATION

This study was initially motivated by Journel and Deutsch (1993). Journel and Deutsch performed a case study with waterflood simulation performed on stochastic realizations for heterogeneous porous media. They compared the responses obtained from three random function models used in stochastic simulations with different spatial entropy. They observed that the responses have a larger variance for random function model with larger entropy, and that the responses have smaller variance for the opposite case. Both of the responses, effective permeability of the heterogeneous porous media and the late breakthrough time, showed the same behavior. Journel and Deutsch concluded that

> "Maximum entropy of the random function model does not entail maximum entropy of the response distributions; in fact, the contrary is observed for the flow performance predictions studied above."

Journel and Deutsch show that applying the principle of insufficient reason (or the principle of maximum entropy) to the states of nature in a decision making problem is not a rational approach and could lead to under-representing the degree of uncertainty in the outcomes of the decision.

A real-world example of large variability in oil production because of heterogeneity is the Holstein field in the Gulf of Mexico (Appendix A).

1.3 OBJECTIVE

The objective of this dissertation is to formulate and study a decision-based method to establish a non-informative prior probability distribution for decision analysis. This dissertation is to evaluate the hypothesis that the decision-based method is rational, practical, and that it is an improvement over the principle of insufficient reason. The evaluation will be made on the following basis:

1. Does the decision-based method produce rational and reasonable results?

2. Can the decision-based method be applied consistently and practically so that a decision maker always gets the same starting point for the same problems?

3. Does the decision-based method satisfy the axioms of decision theory?

1.4 APPROACH

Approach to satisfy the objective is as follows:

1) State and explain the decision-based method to establish unbiased and rational starting point for decision making.

2) Study the decision-based method considering rationality, consistency and defensibility.

3) Develop a practical algorithm for assigning the non-informative prior probabilities to the states of nature.

4) Apply the decision-based method to practical decision making problems in engineering. The three example problems involve oil recovery from heterogeneous porous media and making decisions about oil production facilities. The examples will illustrate the process to establish non-informative prior probabilities and also highlight practical implications of the decision-based method.

1.5 Organization of Dissertation

In Chapter 2, basic concepts and definitions related to Bayesian decision analysis under complete ignorance will be introduced. The purpose of this presentation is to define a clear context for this research and to provide a background on decision theory. Chapters 3 and 4 are about the principle of insufficient reason, the principle of insufficient reason and its extensions, to establish non-informative probability distributions. In Chapter 4, challenges in the application of the principle of insufficient reason are addressed. The decision-based method for establishing non-informative prior probabilities is described in Chapters 5 and 6. The rationale and a practical algorithm for implicating the method are provided. In Chapter 7, the decision-based method is applied to practical engineering decision making problems. Conclusions about the decision-based method for non-informative priors are provided in Chapter 8. In appendices, the result of history matching of oil well production in the Holstein (Appendix A), a computer code for the algorithm of decision-based method (Appendix B), a derivation of modified tank model (Appendix C), and a computer code for the two-dimensional grid simulator of oil production used in the dissertation (Appendix D) are provided.

Chapter 2. Bayesian Decision Theory and State of Complete Ignorance

2.1 INTRODUCTION

Decision theory is originated from the observation of human behavior when making a decision and has been developed to understand how to make a rational decision (Neumann and Morgenstern, 1944; Luce and Raiffa, 1957). The questions that decision theory tries to answer are about how to determine the best alternative (decision criteria), how to incorporate uncertainty, and how to interpret existing information such as statistical data and experimental results. Science, management, and economics are involved in decision theory and its application covers most of the fields of study.

Bayesian decision theory is a division of statistical decision theory. As shown in its own name, Bayesian decision theory is based on the Bayesian definition of probability (probability as a degree of belief) in contrast to frequency probability (Nagel, 1939; Jaynes, 1968; Vick, 2002). The probabilities are used to illustrate the uncertainty involved in decision parameters and accordingly, decision outcomes.

The objective of this chapter is to clarify concepts used as background.

2.2 RISK-BASED DECISION MAKING

2.2.1 Decision and Utility

Decision making is the process of finding the most preferred alternative among a set of alternatives (or options, actions, strategies). The process is a sort of optimizing based on maximizing satisfaction or on minimizing dissatisfaction. The optimized solution is the most preferred alternative, which has the largest measure of satisfaction. Consequence is defined as the measure of satisfaction made by the outcome of a decision.

Consequence in engineering decision is normally expressed in terms of monetary value, such as net present value. A widely used and generic representation of consequence is utility, a quantitative measure that is ordered so that a large value means greater preference by the decision maker.

The modern theory of utility was developed by Neumann and Morgenstern (1944). A utility function relates the consequence of a decision to a variety of attributes affecting the consequence, such as monetary value. Figure 2.1 shows three example utility functions expressing different attitudes toward changes in an attribute. If a decision maker is risk aversive, a utility function has a steeper slope where the attribute is least preferred. Conversely, a utility function for risk affinitive decision maker has a steeper slope where the attribute is most preferred.

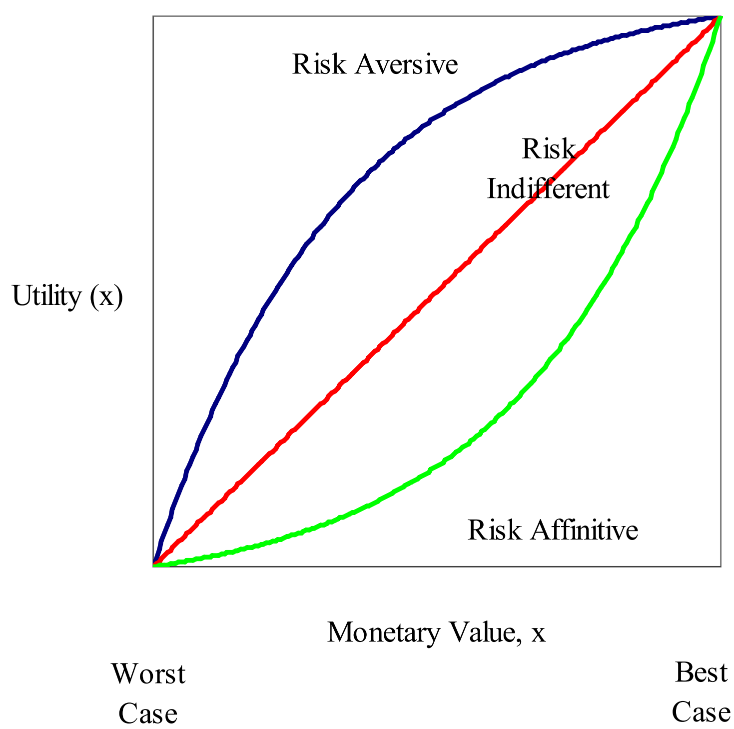


Figure 2.1 Example of utility functions

2.2.2 Risk, Uncertainty, and Probability

In deterministic view point, a decision alterative produces a known consequence, which represents the satisfaction or preference for a decision maker. Deterministic decision making is simply a comparison between the consequence from each decision alternative and determining the alternative with the most favorable consequence. A deterministic approach is not practical in decision making problems in the real world because of the lack of understanding how the state of the world is and how it works as well as the imperfect knowledge about which the future would be. In real world, there are multiple possible consequences because there is a possibility of success (or gain) and a possibility of failure (or loss).

The risk is the possibility of loss. Risk comes from the uncertainty of what the outcome of an alternative would be. Uncertainty can be classified into various sources (Joint Committee on Structural Safety Probabilistic Model Code 2001): intrinsic physical or mechanical uncertainty, model uncertainties, and statistical uncertainties. The first type corresponds to aleatory (Type 1) and represents inherent natural variability. Model uncertainties of inherent actual variability are due to imperfect or imprecise models and statistical uncertainties are due to a limited amount of information about inherent variability. Both model and statistical uncertainties are categorized into epistemic (Type 2) uncertainties. All types of uncertainties should be taken into account in decision analysis (Faber, 2005).

Probability theory provides a method to describe uncertainty mathematically. How likely an uncertain event will occur is represented by probability density function (PDF) for continuous variables or by probability mass function (PMF) for discrete variables.

2.2.3 Risk-Based Decision Making Problems

Risk-based decision making is equivalent to decision making under uncertainty. Decision making has three components: a set of alternatives, a set of possible states of nature (states of the world, or scenarios), and consequences associated with an alternative and a state of nature. The following sections will provide a background on the theory of making decisions.

2.2.3.1 Decision criteria

Kelsey and Quiggin (1992) summarized the historical and theoretical background and major differences of these decision criteria. For symbolic description, suppose we have a decision making problem with *n* alternatives (A_1, A_2, …, A_n), *m* states of nature (s_1, s_2, …, s_m), and consequences in utility (u_{ij}, i=1,2,…, *n*, j=1,2,…,*m*).

Historical criteria include the maximin, minimax, and Hurwicz criteria. In the maximin criterion, the index for A_i is assumed to the minimum among possible consequences, min($u_{i,1}$, $u_{i,2}$, …, $u_{i,m}$). This assignment is indexing the worst state for each alternative. A decision maker chooses the alternative with maximum index as an optimal. The minimax risk criterion is suggested by Savage (1951). This criterion begins with a definition of loss (risk, or regret in some literature): L_{ij}=max(u_{i1}, u_{i2}, …, u_{im})-u_{ij} for a given s_j. The loss, L_{ij} replaces the utility consequence, u_{ij} in the decision making problem. The index imposed to each alternative is determined by taking the maximum loss possible, max(L_{i1}, L_{i2}, …, L_{im}) for a given alternative. The best alternative is chosen by picking the one with the minimum index, in other words, the one with minimum risk. While maximin and minimax risk criteria are based on the worst case, the Hurwicz criterion introduces the other way to avoid the pessimistic viewpoint. Hurwicz (1951) introduced a new index called the pessimism-optimism index, which is symbolized with α. The index is assigned to each alternative, A_i and equal to $\alpha \times \min(u_{i1}, u_{i2}, \ldots, u_{im}) + (1-\alpha) \times \max(u_{i1}, u_{i2}, \ldots, u_{im})$. If $\alpha=1$, Hurwicz criterion is the same with maximin utility criterion and if $\alpha=0$, it is the same with maximax utility criterion.

All decision criteria above are based on human behavior about the risk in decision making. The attitude toward risk was pessimistic for maximin and minimax risk criteria and controllable for Hurwicz criterion. Those criteria do not incorporate probabilities on states of nature so that a decision maker ignores the likelihood of occurrence of each state. The most widely accepted and defensible criterion is based on the maximum expected utility. This criterion incorporates the probabilities on states of nature, $P(S_j)$, $j=1, 2, \ldots, m$. The expected utility for A_i is calculated by making a summation of the products, $P(S_j) \times u_{ij}$ for all j states. The basis for using maximum expected utility is provided by many authors, including Bernoulli (1738), Neumann and Morgenstern (1944), and Kelsey and Quiggin (1992), and it is supported by a set of axioms that have been widely accepted as rational and reasonable. For example, Luce and Raiffa (1957) expressed these axioms in a set of nine. The preferred decision alternative is the one with the maximum expected utility. In this study, maximum expected utility criterion is accepted and used.

2.2.3.2 Decision matrices and trees

A symbolic description for decision making can be illustrated by a decision matrix, as shown in Table 2.1. The decision matrix is a convenient tool to summarize a simple decision making problem with alternatives, states of nature, and consequences.

A decision tree provides a graphical way to illustrate a decision making problem. It provides a systematic way to integrate decision components: the decision alternatives, the uncertainty (in terms of the states of nature and the probabilities assigned to those states), and the consequences (in terms of utility) in decision making problems. Figure 2.2 is an example of decision tree. A square node is called a decision node where a decision maker makes a decision, and a circular node is a chance node which represents the uncertainty. Therefore, branches stemming from the decision node are alternatives, A_1 through A_n, and those from the chance node are possible states of nature, S_1 through S_m. Probabilities are assigned to each state of nature. The expected utility for an alternative is equal to the summation of the product of probabilities and consequences.

The tree structure enables to illustrate not only a complicated decision structure, but also a causal relationship between events. A present decision comes first and then a future decision follows. A certain event or decision affects events along the pathway so that the probability of a following event should be conditional to a leading event.

Table 2.1 Example of a decision matrix

		States of Nature			
		S_1	S_2	…	S_m
Decision Alternatives	A_1	u_{11}	u_{12}	…	u_{1m}
	A_2	u_{21}	u_{22}	…	u_{2m}
	:	:	:		:
	A_n	u_{n1}	u_{n2}	…	u_{nm}
Probabilities		$P(S_1)$	$P(S_2)$		$P(S_m)$

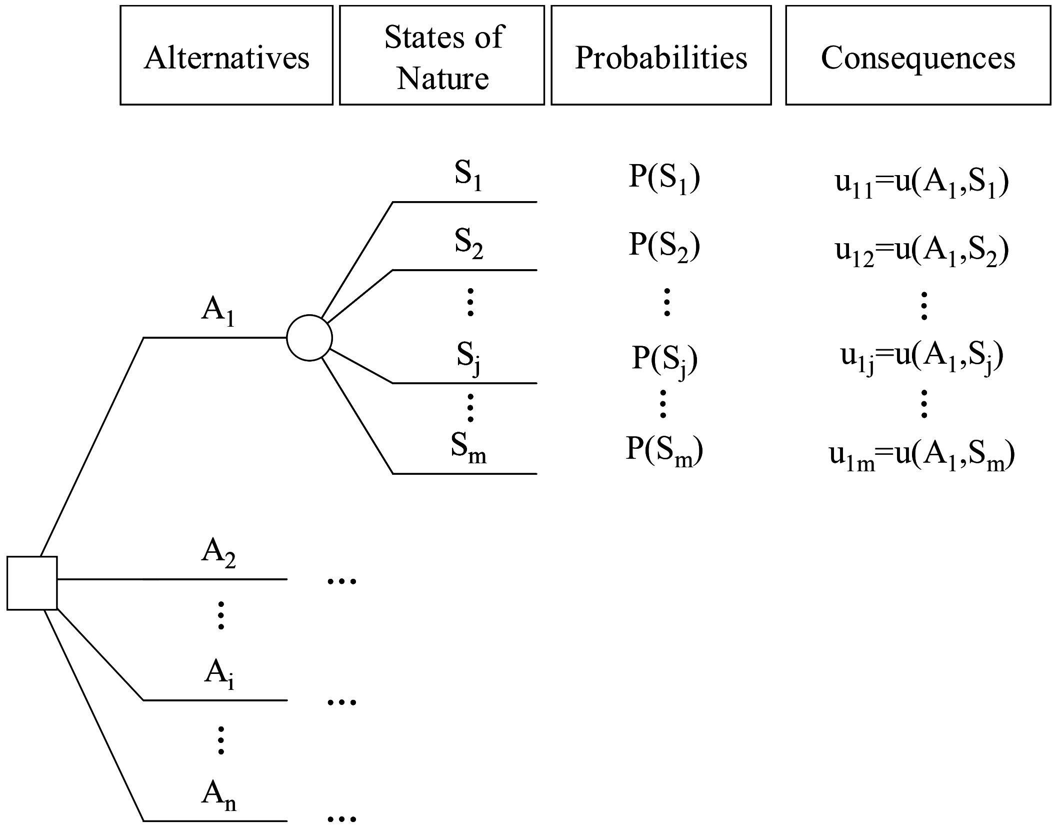


Figure 2.2 Basic decision tree

2.2.3.3 Prior, posterior, and preposterior decision

The basic decision structure shown in Figure 2.2 is called a prior decision in Bayesian decision theory. The qualifier "prior" refers to before any new information becomes available. There exist some variations from the prior decision according to the availability of new information on states of nature. A posterior decision, shown in Figure 2.3, is an updated decision tree based on incorporating new information. A posterior decision making problem is basically the same with prior decision making problem. The new information is used to update probabilities on states of nature, $P(S_i|\text{Information, I})$. A preposterior decision, shown in Figure 2.4, is a decision to consider the potential value that new information might have before the information is obtained. The decision alternatives for preposterior analysis are a Go-No go decision about purchasing new information. In engineering examples of new information includes database, site exploration data, sampling, or modeling. The root node in Figure 2.4 is the decision about purchasing new information. If a decision make decides to purchase new information, the outcome of having new information should follow the branch stems from the root in the decision tree. The outcome is represented by a chance node because there is uncertainty in the outcome of the information. This uncertainty is caused by the fact that a decision maker is making a decision without knowing what the new information would be. Each branch of the possible outcome of the new information, I_i, is followed by the decision structure of posterior analysis with updated probabilities on states of nature.

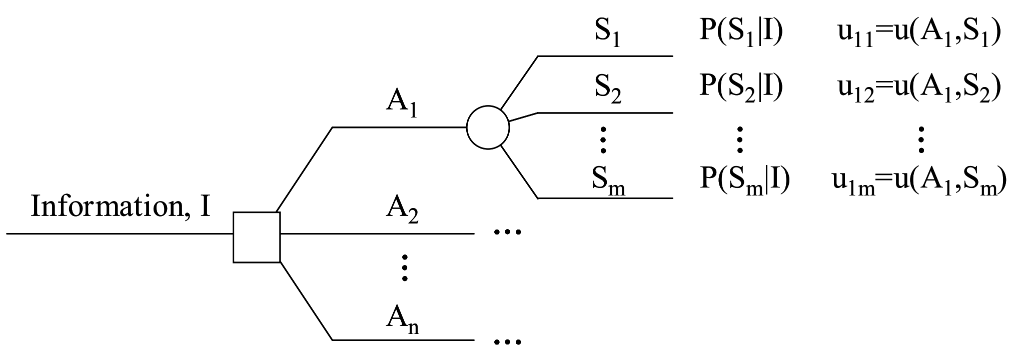


Figure 2.3 Decision tree for posterior analysis

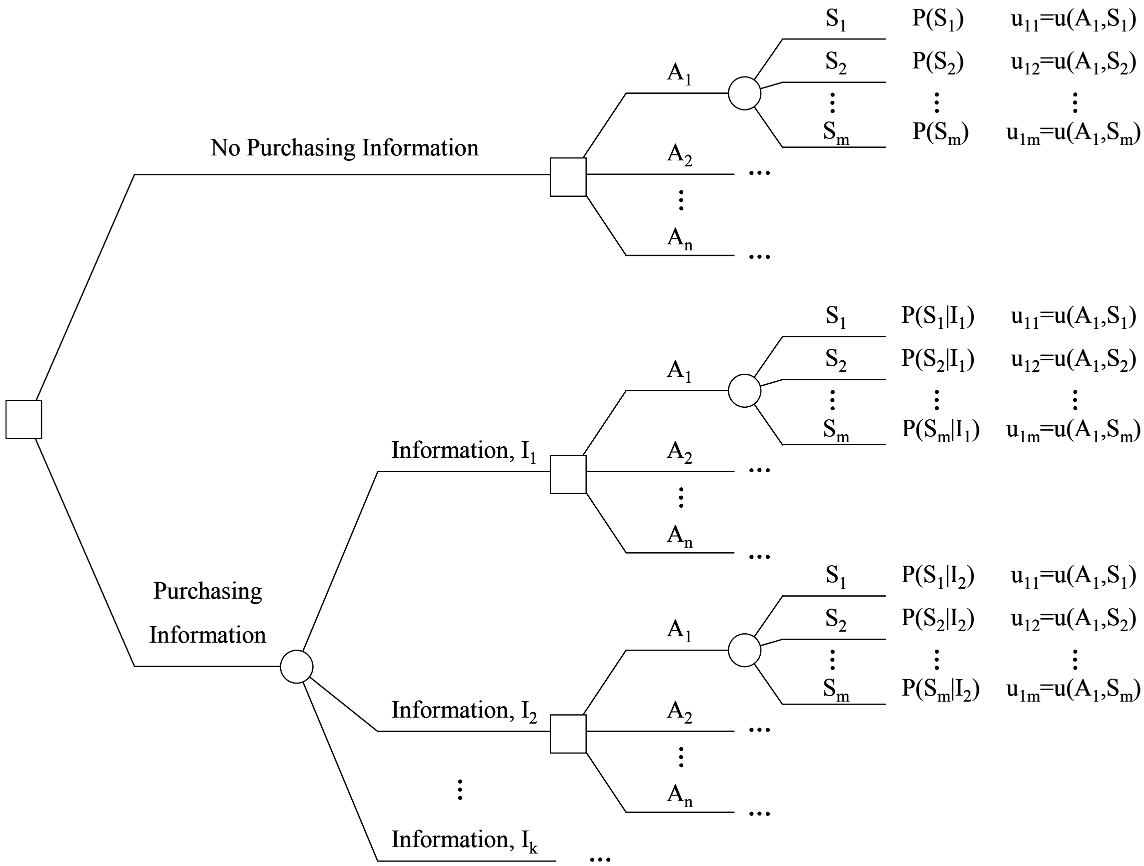


Figure 2.4 Decision tree for preposterior analysis

These three types of decision trees can cover various practical decision making problems. Faber (2005) categorizes engineering problems as decision making problems, as shown in Table 2.2. Each engineering decision making problem has a corresponding decision structure among the three types. As well as to make a decision, preposterior decision analysis is used to assess value of information (VoI, VI) or value of technology. The value of information is equal to the difference between the expected utility of two decision alternatives, "No purchasing information" and "Purchasing information." If the information is totally useless, that is, the information does not help to infer how likely each state of nature would be at all, the VoI is equal to zero. If the information contains more information on the likelihood of the states of nature that possibly changes the prior decision, VoI would increase. The upper bound of VoI is called the value of perfect information (VPI). In the same manner with VoI, the VPI is the difference in the expected utilities between "No purchasing information" and "Purchasing perfect information." Perfect information means that the information source is able to inform which of states of nature would occur with certainty. The preposterior decision tree in Figure 2.4 can be modified for estimating VPI, as shown in Figure 2.5. The conditional state probabilities, $P(S_i|I_j)$, in a VPI analysis have values of 0 when $i \neq j$ and 1 when $i=j$.

Table 2.2 Categorization of engineering problems as decision making problems (based on Faber, 2005)

Decision theoretical problem	Engineering problem
Prior	• Risk assessment for verification • Design and strengthening optimization • Calibration of risk acceptance criteria • Calibration of code formats (γ, ψ)
Posterior	• Reliability updating for service life extensions • Reliability updating for re-qualification
Preposterior	• Planning of collection of information (tests, experiments, and proof load levels) • Inspection and maintenance planning

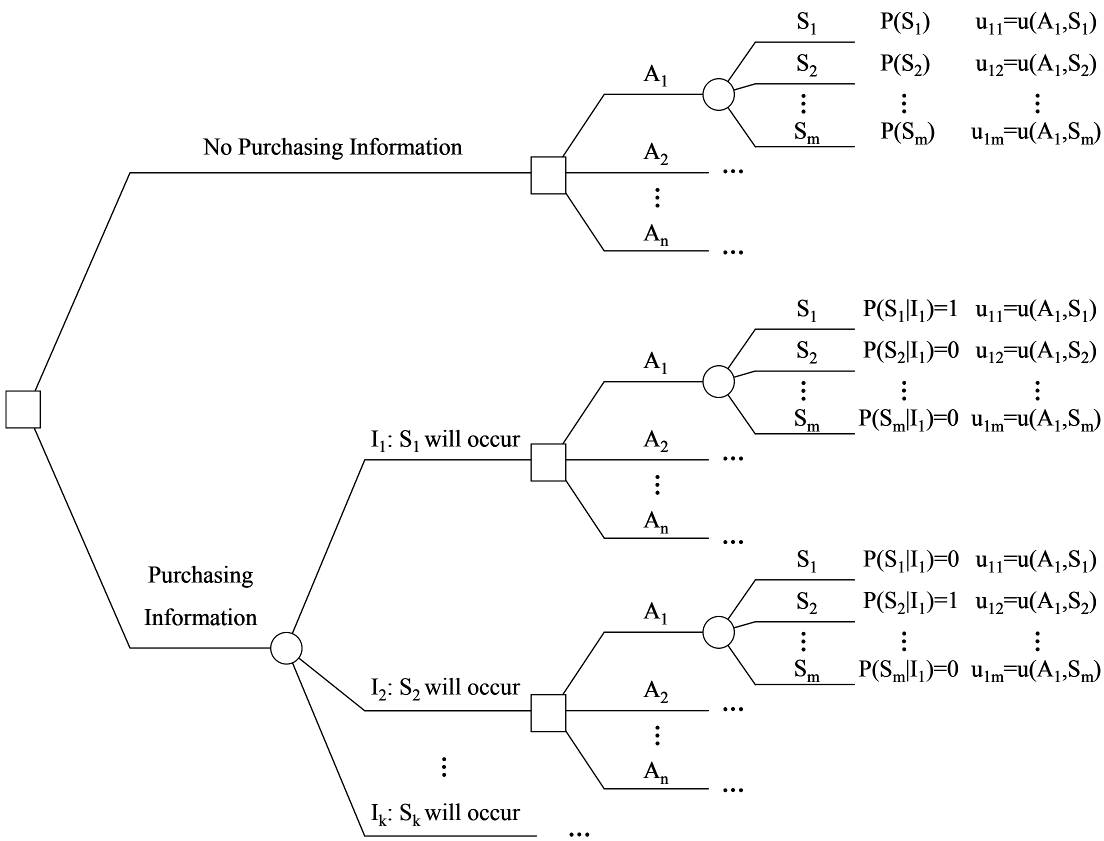


Figure 2.5 Decision tree for the value of perfect information analysis

2.3 BAYES' THEOREM

2.3.1 Bayes' Theorem

Bayes (1763) theorem relates a marginal and conditional probability, as shown in Equation 2.1. P(A) is a prior probability distribution, P(A|B) is a posterior probability distribution, and P(B|A) is a likelihood function that relates those two probability distributions.

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \tag{2.1}$$

Bayes' theorem can be interpreted as a learning process if the event A is assumed to be the states of nature and B is available information. Bayes' formulation can be expressed, as shown in Equation 2.2. In this viewpoint, P(A) is equivalent to a probability distribution on the states of nature, $P(S_i)$ before having the new information and P(A|B) is an updated probability distribution by the new information, $P(S_i|\text{Information})$. The information may include both objective and subjective information, for example, field and lab test results, experts' opinions, and knowledge from a database.

$$\begin{aligned} P(S_i \mid \text{Information}) &= \frac{P(\text{Information} \mid S_i) P(S_i)}{P(\text{Information})} \\ &= \frac{P(\text{Information} \mid S_i) P(S_i)}{\sum_{j=1}^{\text{Number of States}} P(\text{Information} \mid S_i) P(S_i)} \end{aligned} \tag{2.2}$$

Since Bayes' theorem was formulated, it has been questionable that which should be the prior probability distribution, $P(S_i)$ in Equation 2.2. Berger (1985), Kass and Wasserman (1994), and Carlin and Louis (1996) summarized and discussed about the methods to establish the prior probability distribution. It is an important and critical question because a different prior probability distribution may yield a different posterior probability distribution. Suppose we have two cases that have different prior probability distributions, as shown in Figure 2.6, but the same likelihood function in Figure 2.7. By Bayes' theorem, the updated probability distributions for both cases can be calculated. As shown in Figure 2.8, the posteriors are different from each other.

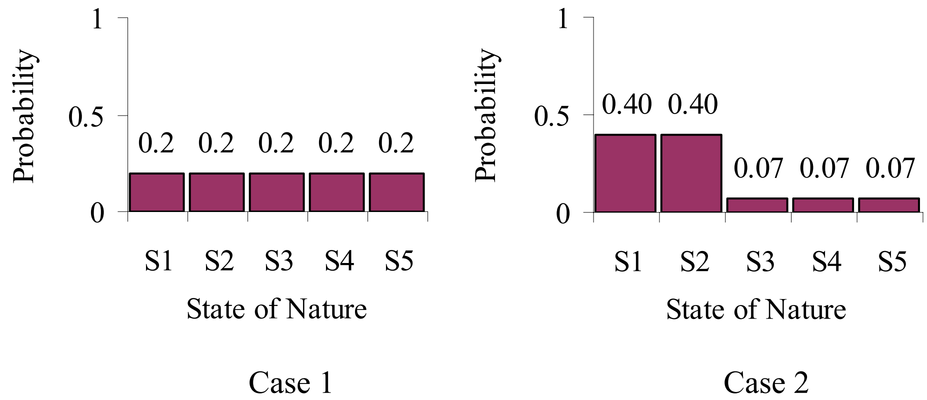


Figure 2.6 Two different prior probability distributions

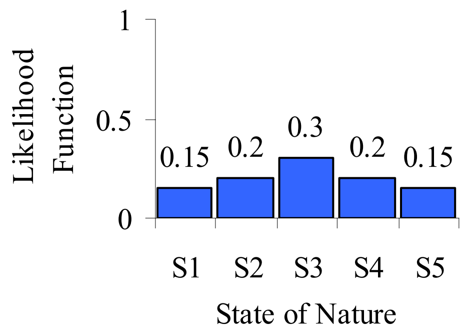


Figure 2.7 Likelihood function from available information

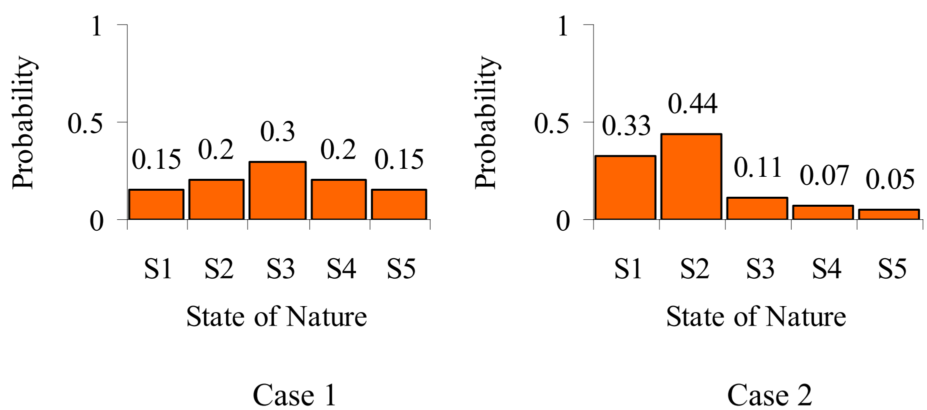


Figure 2.8 Two different posterior probability distributions from the different prior and the same information

2.3.2 Bayesian Approach in Decision Making

The Bayesian approach provides a method to incorporate new information into the probability distribution that takes a previous knowledge into account. The resultant probability distribution, which is called posterior distribution, varies with what the information implies about an uncertain variable. In decision making problems, the uncertain variables are the states of nature and the probabilities on the states in prior decision analysis can be interpreted as prior probabilities. If new information comes available such as an information branch in posterior or preposterior decision trees (Figures 2.3 and 2.4), the probabilities at the terminal branches requires Bayesian framework to reflect the new information. In this Bayesian framework, the conditional probabilities to certain information are posterior probabilities.

Figure 2.9 illustrates a preposterior Bayesian decision example. For the alternative of not purchasing additional information, the probability mass function on an uncertain variable, x is assumed to be probabilities at the top of Figure 2.9. For the no purchasing alternative, this probability distribution is used to optimize the following set of alternatives, A_1 and A_2. For the other alternative of purchasing information, there are two possible outcomes from the information. Because the probability distributions for x at each terminal are conditional to the leading branches (events), each probability distribution following certain information is updated through Bayesian updating. Info. 1 reveals that the small value of x is more likely. This new information and the previous knowledge are incorporated to yield the updated probability distribution shown in the second to the top. In the same manner, posterior probability distributions regarding Info. 2 are established.

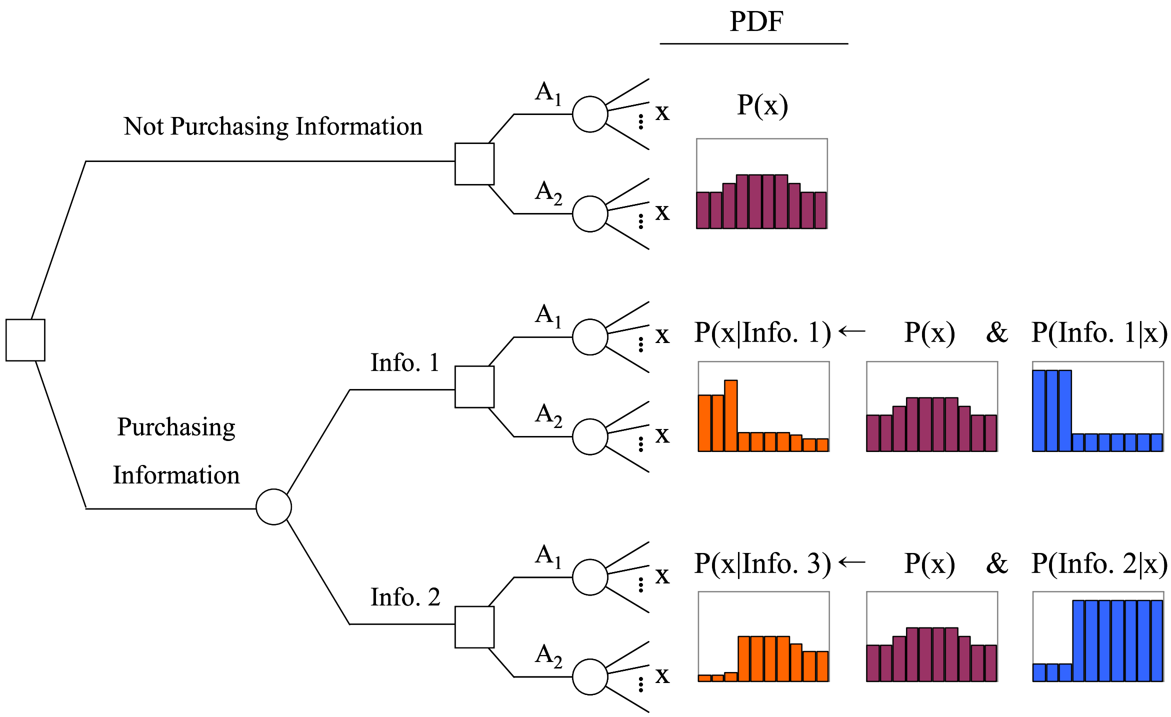


Figure 2.9 Probability mass functions in a Bayesian preposterior decision analysis

The importance of the prior probability distribution in decision making can be illustrated with the example in a previous section with two different priors shown in Figure 2.6, the same likelihood function in Figure 2.7, and the two subsequent posteriors in Figure 2.8. Suppose we assign a decision framework by the decision matrix in Tables 2.3 and 2.4. Prior probabilities as in Case 1 have the second alternative, A_2 preferred to the first alternative, A_1 because the expected utilities for both alternatives, A_1 and A_2 are 3.5 and 5.2, respectively. For prior probabilities as in Case 2, A_1 is preferred to A_2 because the expected utilities are 7.6 and 1.9, respectively. This example shows that different prior may produce different posterior and change an optimal decision.

Table 2.3 Decision matrix for Case 1

		States of Nature					Expected Utility
		S_1	S_2	S_3	S_4	S_5	
Decision Alternatives	A_1	10	10	0	0	0	3.5
	$\mathbf{A_2}$	0	0	8	8	8	**5.2**
Probabilities		0.15	0.2	0.4	0.2	0.15	

Table 2.4 Decision matrix for Case 2

		States of Nature					Expected Utility
		S_1	S_2	S_3	S_4	S_5	
Decision Alternatives	$\mathbf{A_1}$	10	10	0	0	0	**7.6**
	A_2	0	0	8	8	8	1.9
Probabilities		0.33	0.44	0.11	0.07	0.05	

2.4 COMPLETE IGNORANCE

2.4.1 State of No Knowledge

The state of complete ignorance is the state of maximum uncertainty. The state is also called deep uncertainty because a decision is made under no knowledge of correlation and probability distributions of the variable (Lempert *et al.*, 2006). It should be noted that the terms, uncertainty and risk sometimes are used in different meanings.

In some literature, "risk" refers the situation that a decision maker knows the probability distribution for states of nature and "uncertainty" means that even the probability distribution is unknown (Luce and Raiffa, 1957). In this study, the state of complete ignorance is equivalent to the state of "uncertainty" in Luce and Raiffa's terminology. The condition of complete ignorance does not mean that the decision maker is totally blind to the states of nature. The decision maker knows a set of possible states of the nature and consequences corresponding each state for a given alternative, but does not know how likely each state would be.

2.4.2 Non-Informative Probabilities

The likelihood of each state of nature under complete ignorance is called a non-informative probability. Non-informative prior probabilities correspond to a prior decision making problem under complete ignorance. For prior decision making problems with information, the probability distribution for states of nature is $P(S_i|\text{Information})$ in Equation 2.2. The prior probability, $P(S_i)$, in a Bayesian framework is equivalent to $P(S_i|\text{Sample space or Set of all possibilities})$. $P(S_i|\text{Sample space})$ is referred as just "prior", "gentle prior" (Pratt, 1995), or "non-informative prior". It will be called a non-informative prior probability distribution in this study.

The non-informative prior probability distribution is an important factor in decision making. The reason is that the non-informative prior probability distribution is a starting point for a decision. The non-informative prior affects a decision because it is a basis of posterior probability distribution, expected utilities of each decision alternative, and ultimately a final decision. All decisions that include information are actually posterior decisions. The influence of the non-informative prior is shown with a simple example in the previous section (Figures 2.5, 2.6, and 2.7 and Tables 2.3 and 2.4).

Because of this importance, it has been argued for many years (centuries) about what is the most logical and reasonable method to establish a non-informative probability distribution. Unfortunately, there has been no satisfactory answer fulfilling mathematical, theoretical, and rational requirements. The following two chapters will explain and discuss the principle of insufficient reason for establishing a non-informative prior probability distribution.

Chapter 3. Non-Informative Probabilities and the Principle of Insufficient Reason

3.1 INTRODUCTION

The objective of this chapter is to review principle of insufficient reason to obtain a non-informative probability distribution. This review addresses the most widely used method, the principle of insufficient reason and its practical implementation through the principle of maximum entropy.

3.2 PRINCIPLE OF INSUFFICIENT REASON

In conventional decision making problems or probability assessment problems with no information, people intuitively assign equal probabilities to the states of nature when there is no evidence or belief that any state is more likely than the rest of states. This idea is originated by Bernoulli and Laplace at the early age of probability theory, and is generally called the principle of insufficient reason. Keynes (1957) preferred to name it the principle of indifference. This principle supports the use of rectangular or uniform probability distributions in a situation of complete ignorance. The principle is also widely accepted in various fields of research (Walstrom *et al*., 1967; Benjamin and Cornell, 1970; Ang and Tang, 1975; Jensen *et al*.,1997; Seigneur *et al*., 1999; Rechard and Tierney, 2005).

Assigning the same probabilities for every state of nature is the main idea of the principle of insufficient reason. This idea is called equiprobability. Laplace (1814) restricted the application of equiprobability within equipossible events. From this equipossibility concept, the principle of insufficient reason assigns the same probabilities on the events of the same possibility. The notion of equipossibility seems vague to define clearly. It is evident that the challenge in defining a sample space for the application of the principle of insufficient reason has been noticeable from the beginning of probability theory. In Section 3.4, other efforts made for defining the sample space for states of nature where the principle of insufficient reason is applied to will be presented.

3.3 PRINCIPLE OF MAXIMUM ENTROPY

3.3.1 Entropy as a Measure of Uncertainty

The concept of entropy was introduced in thermodynamics to describe the second law of thermodynamics. Entropy describes the degree of disorder or the state of the dispersed energy. Entropy can also be used to measure how much a probability distribution is diffused and accordingly, for how much uncertainty the probability distribution has.

This use of entropy is supported by information theory. Shannon introduced a concept of entropy into information theory (Shannon and Weaver, 1949). The entropy, which had been used as a measure of disorder in thermodynamics, was proved as a reasonable measure of uncertainty. Their definition in the measure of uncertainty, H begins with the case where a set of *n* possible events has probabilities of occurrence, p_1, p_2, …, p_n. The requirements for the measure of uncertainty are:

1. H should be continuous in the p_i.

2. If all p_i are equal, in other words, p_i is equal to 1/n, H should be a monotonic increasing function of n.

3. If an uncertainty is broken down into following uncertainties, the original H should be the weighted sum (expected value of sub-uncertainty) of the individual values of H.

The meaning of requirement 2 is that the uncertainty increases if the number of states of nature increases. It is reasonable because the increment of the number of states becomes additional unknowns mathematically, and represents a higher order of uncertainty to describe the state. Requirement 3 is about the situation where a certain possible state is broken into sub-cases. For consistency, a measure of uncertainty should have the same value for both of the uncertainty structures shown in Figure 3.1 and Equation 3.1.

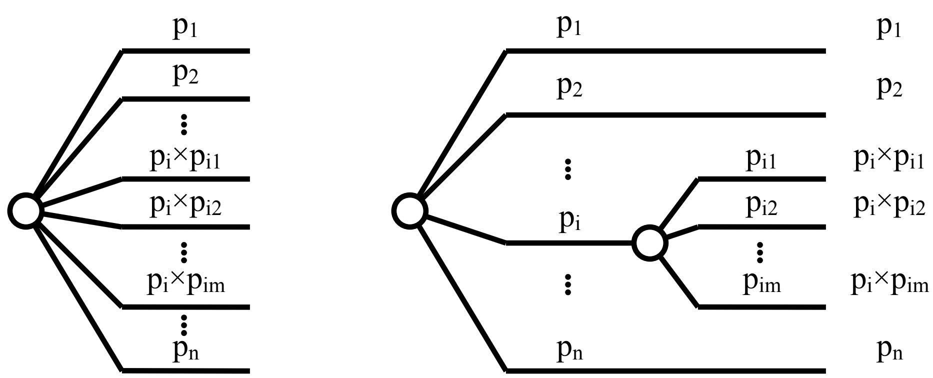


Figure 3.1 Decomposition of uncertainty structure

$$
\begin{aligned}
&H\left(p_1, p_2, \ldots, p_{i-1}, p_i \times p_{i1}, p_i \times p_{i2}, \ldots, p_i \times p_{im}, p_{i+1}, \ldots, p_n\right) \\
&\quad = H\left(p_1, p_2, \ldots, p_n\right) - p_i H\left(p_{i1}, p_{i2}, \ldots, p_{im}\right)
\end{aligned}
\tag{3.1}
$$

From the derivation based on the properties, a mathematical definition of the entropy as a measure of uncertainty, symbolized by H, is shown in Equation 3.2. The Equation 3.2 is for discrete probability distributions. For continuous probability distributions, the measure of uncertainty is given by relative entropy (see Section 4.5.1) because the value of the measure in Equation 3.2 approaches infinity in the limiting process of transformation from the discrete to the continuous case (Baker, 1990).

$$H = -\sum_{i=1}^{n} p_x(x_i)\ln(p_x(x_i)) \tag{3.2}$$

Equation 3.1 can be used to illustrate the requirement 2, as shown in Figure 3.2. The definition of H satisfies the requirement 3, as shown in Equation 3.3.

$$\begin{aligned}
&H(p_1, p_2, \ldots, p_{i-1}, p_i \times p_{i1}, p_i \times p_{i2}, \ldots, p_i \times p_{im}, p_{i+1}, \ldots, p_n) \\
&\quad = -\sum_{j\neq i}^{n} p_j \ln(p_j) - p_i \sum_{j=1}^{m} p_{ij} \ln(p_i \times p_{ij}) \\
&\quad = -\sum_{j\neq i}^{n} p_j \ln(p_j) - p_i \sum_{j=1}^{m} p_{ij} \ln(p_i) - p_i \sum_{j=1}^{m} p_{ij} \ln(p_{ij}) \\
&\quad = -\sum_{j\neq i}^{n} p_j \ln(p_j) - p_i \ln(p_i) - p_i \sum_{j=1}^{m} p_{ij} \ln(p_{ij}) \\
&\quad = -\sum_{j}^{n} p_j \ln(p_j) - p_i \sum_{j=1}^{m} p_{ij} \ln(p_{ij}) \\
&\quad = H(p_1, p_2, \ldots, p_n) + p_i H(p_{i1}, p_{i2}, \ldots, p_{im})
\end{aligned} \tag{3.3}$$

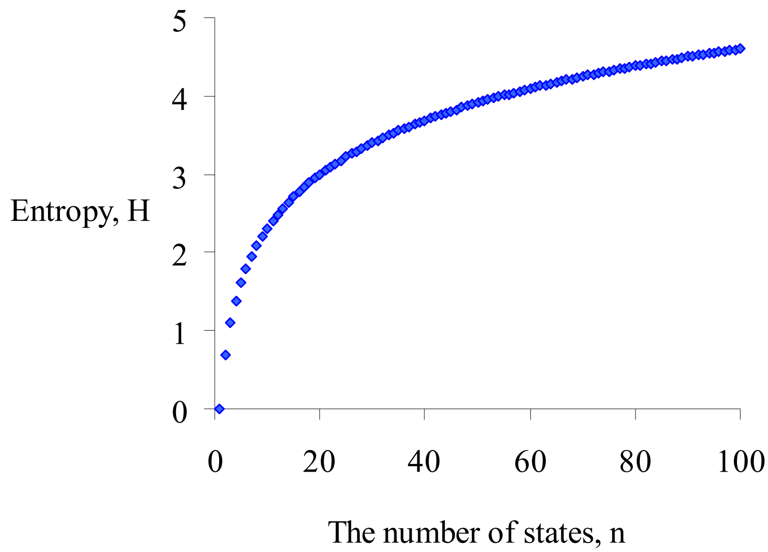
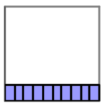
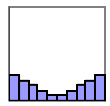
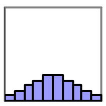
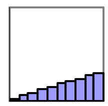
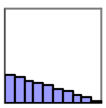
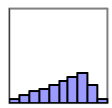
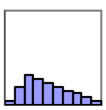
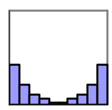
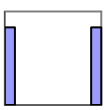
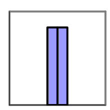


Figure 3.2 Entropy versus the number of states for illustrating the second requirement for the measure of uncertainty in Shannon and Weaver (1949)

Estimated only with the probabilities, the entropy does not depend on the values of a variable. Therefore, an entropic approach is applicable whether a variable is numeric or descriptive. This feature is useful if the uncertainty model is based on the discrete probability mass function (PMF) where each class has a corresponding range of a variable.

Table 3.1 shows the entropy for various PMFs. The number of classes is 10 and each class is assumed to have a value between 1 and 10 in ascending order. In addition, for comparison purpose, the variance of each PMF is also shown in Table 3.1.

Table 3.1 Comparison of variance and entropy as a measure of uncertainty

Case	PMF	Entropy (Rank)	Variance (Rank)	Case	PMF	Entropy (Rank)	Variance (Rank)
1		2.303 (1)	8.25 (4)	6		2.183 (2)	11.583 (3)
2		2.183 (2)	4.917 (7)	7		2.151 (4)	6 (5)
3		2.151 (4)	6 (5)	8		2.135 (6)	4.853 (8)
4		2.135 (6)	4.853 (8)	9		1.936 (8)	14.573 (2)
5		0.693 (9)	20.25 (1)	10		0.693 (9)	0.25 (10)

There are general remarks for the results in Table 3.1. The variance increases with the probability distribution far from the average. This is why there exists PMFs having greater variance than the uniform distribution (Cases 6, 7, and 9). The entropy is the same for PMFs having the same set of probabilities whatever the order of those probabilities. Because the entropy is calculated only by probabilities, the order of the probabilities does not matter. Therefore the concave and convex shaped PMFs have the same entropy as each other (Cases 2 and 6, 3 and 7, 4 and 8, and 5 and 10).

The variance is not necessarily a good measure of uncertainty, specifically when the PMF is concave-shaped to down. Case 5 shows the largest variance but seems very ordered. The PMF of Case 5 is not the uniform distribution, but the distribution of 50-50 chance for both extremes. This probability distribution is the one with less uncertainty rather than the one with the most uncertainty. This limitation is caused by the nature of variance. Variance measures the distance from the average, therefore the concave shaped PDF has more variance than the uniform distribution. While the variance fails to measure the uncertainty in some cases, entropy works well as a measure of uncertainty. The PMF of maximum uncertainty is Case 1, which has a uniform probability distribution. For defining the maximum uncertainty, it is generally appropriate to make use of entropy.

3.3.2 Non-Informative Probabilities from Principle of Maximum Entropy

Tribus (1969) indicated that the probability distribution with maximum entropy and no information (constraints) is the uniform distribution, and that the principle of insufficient reason is a special case of the application of the principle of maximum entropy. The principle of insufficient reason, assigning the same probability or uniform distribution to uncertain parameters, has a mathematical support from the principle of maximum entropy. Jaynes (1957) addressed that the principle of maximum entropy may be regarded as an extension of the principle of insufficient reason.

Chapter 4. Difficulties in the Application of the Principle of Insufficient Reason

4.1 INTRODUCTION

Establishing non-informative prior probabilities based on the principle of insufficient reason is seemingly intuitive to understand and simple to apply. Therefore, it is widely used in estimation, prediction, and decision making problems (Walstrom *et al*., 1967; Benjamin and Cornell, 1970; Ang and Tang, 1975; Jensen *et al*., 1997; Seigneur *et al*., 1999; Rechard and Tierney, 2005). In this study, the focus is narrowed to non-informative probability distributions for decision making problems.

In this chapter, three major difficulties with the principle of insufficient reason when applied to decision making problems will be discussed. The first difficulty is that the non-informative probabilities can inform or bias a decision. The second difficulty is that this principle does not provide a consistent set of non-informative probabilities in practice. The third difficulty is that this principle is not consistent with axioms of decision theory in the case of complete ignorance. The first difficulty is a unique notion of this research while the second and the third ones already have been revealed by many literatures.

4.2 BIAS IN DECISION

The principle of insufficient reason assigns the same probabilities to the states of nature. The fundamental idea of the principle of insufficient reason is that a complete ignorance is equivalent to the most uncertainty in states of nature. However, this idea can mislead a decision making into a biased decision. Suppose a decision maker has a decision making problem with two alternatives and three possible states of nature, as shown in Figure 4.1. The principle of insufficient reason would mean that the three states have the same probability of ⅓.

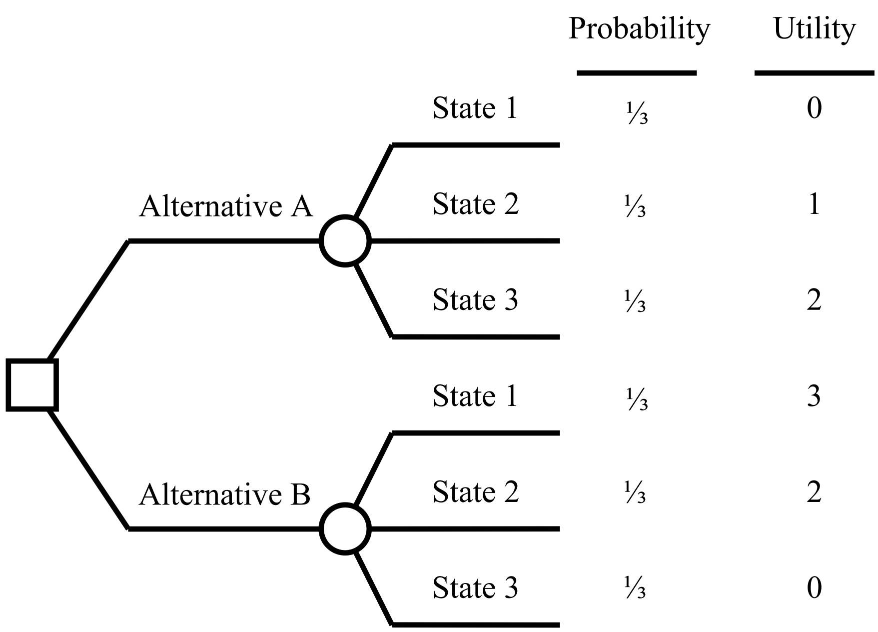


Figure 4.1 Decision tree for simple example

The preference in decision alternatives for each state is shown in Table 4.1. Alternative B is preferred to Alternative A in States 1 and 2, while Alternative A is preferred in State 3. Therefore, the probabilities that one decision alternative is preferred to the other can be calculated, as shown in Figure 4.2. The probability that Alternative A is preferred to Alternative B is equal to the probability of State 3, ⅓. The probability that Alternative B is preferred to Alternative A is ⅔, as P(State 1)+P(State 2). Alternative B is more likely to be preferred than the other because P(Alternative A> Alternative B) is less than P(Alternative A< Alternative B), where “<”, “>”, and “~” denote the inequality or equality in preference. By assigning equal probabilities to states of nature, a decision maker makes Alternative B more likely to be preferred. This principle unintentionally instills the bias in decision because the decision is not considered in assigning the probabilities.

Table 4.1 Preference in alternatives for each state

State	Utility		Preference outcome (Preference in decision alternatives)
	Alternative A	Alternative B	
State 1	0	3	Alternative A < Alternative B
State 2	1	2	Alternative A < Alternative B
State 3	2	0	Alternative A > Alternative B

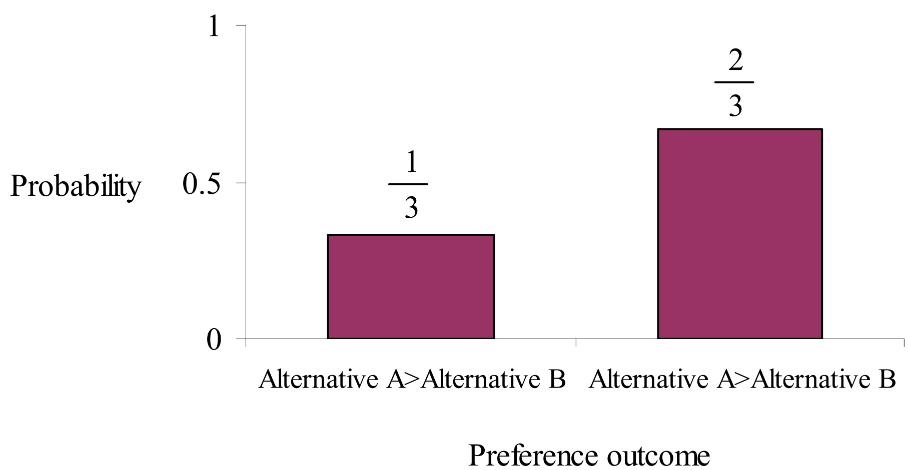


Figure 4.2 Bias in decision

This difficulty with the principle is related to what Journel and Deutsch (1993) concluded. Journel and Deutsch observed that the input for reservoir heterogeneity model with the maximum entropy could provide the response (effective permeability and late breakthrough time) with minimum entropy. In the same manner, a decision maker may have decision alternatives with less uncertainty from assigning the probabilities with the maximum uncertainty such as uniform probabilities to the states of nature based on the principle of insufficient reason. The alternatives with less uncertainty represent the unintentionally uninstalled bias in decision making.

4.3 INCONSISTENCY

The principle of insufficient reason has a drawback in that a probability distribution varies with the designation of the states of nature. Laplace (1814), Luce and Raiffa (1957), Box and Tiao (1973), Shafer (1976), Sinn (1980), and Kass and Wasserman (1994) pointed out this drawback.

4.3.1 Discrete Variable Case: Partitioning Paradox

Shafer (1976) provided a simple example about the partitioning paradox. Scientists have a question about a life near the star Sirius. There are two possible states, A_1 and A_2. Each denotes that a life exists and that no life exists, respectively. The principle of insufficient reason yield ½ on both states. Then, the scientists refined the question on the existence of planets around Sirius. The states of nature in this question consist of three events, $\overline{P}$, $P\overline{A}$, and PA: $\overline{P}$ denotes that a planet does not exist, $P\overline{A}$ denotes that a planet exists but no life, and PA denotes that a planet exists and a life also. Based on the principle of insufficient reason, those three events have the same probability of ⅓ and the probability of life existing has now decreased from ½ to ⅓ even though we have no information in either way of formulating the problem. A_1 is equivalent to the event, PA, and A_2 is to the two events, $P\overline{A}$ and $\overline{P}$. However, $P(A_1) \neq P(PA)$ and $P(A_2) \neq P(\overline{P}) + P(P\overline{A})$. This inequality is called the partitioning paradox.

The inconsistency may also happen for the uncertainty model in decision making problem. The principle of insufficient reason might provide different probability distributions depending on which variable the principle of insufficient reason is applied to. Suppose we have a heterogeneous porous medium, as shown in Figure 4.3. The media consists of two cells and the uncertainty exists in permeabilities of the two cells. The effective permeability, k_{eff} for this composite medium is defined as a harmonic mean, as shown in Equation 4.1. The permeabilities of two parts, k_1 and k_2 might be considered as the input variables, the harmonic mean as a model, and the effective permeability as a response.

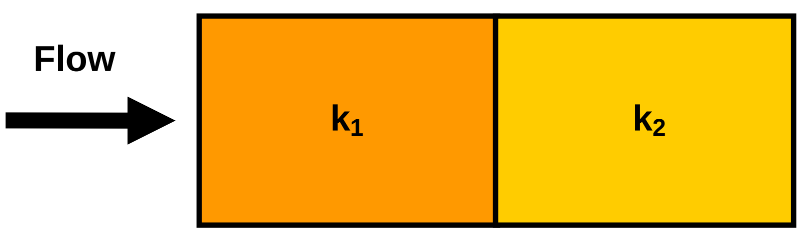
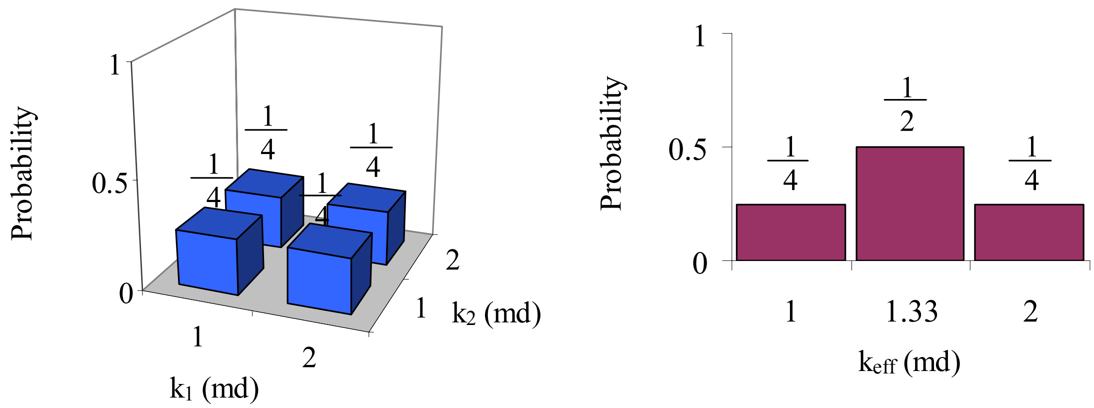


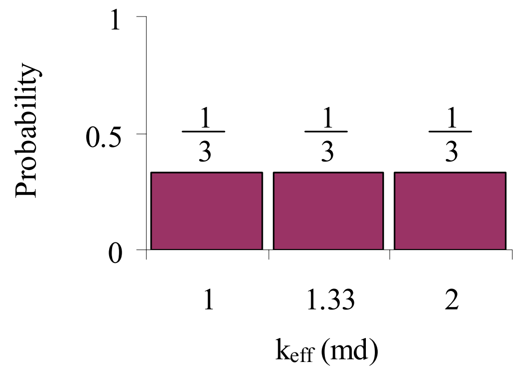
Figure 4.3 Heterogeneous porous medium

$$\frac{1}{k_{eff}} = \frac{1}{2}\left\{\frac{1}{k_1} + \frac{1}{k_2}\right\} \tag{4.1}$$

There are two possible applications of the principle of insufficient reason. In the first way, the principle is applied to the input variables. If there are two possible states in permeability, 1 and 2 (md), the resultant joint probability distribution will be two dimensional uniform distributions, as shown in Figure 4.4a. The combinations of k_1 and k_2 make three possible values of the effective permeability, 1, 1.33, and 2 (md). From the joint PMF of k_1 and k_2, the probabilities of those three states are 0.25, 0.5, and 0.25, respectively. In the second way, the three states might have the same probability when the principle of insufficient reason is applied directly to the response variable. The three bins for the effective permeability have the same probability of ⅓, as shown in Figure 4.4b. The probability distributions for the effective permeability are different from each other. This inconsistency is summarized in Figure 4.4.



(a) Uniform distribution on input variables, k_1 and k_2



(b) Uniform distribution on a response variable, k_{eff}

Figure 4.4 Inconsistent probability distributions caused by the selection of the variable to which the principle of insufficient reason is applied

4.3.2 Continuous Variable Case

The principle of insufficient reason can also result inconsistency for a continuous variable space. If we have a variable, ν and have no information on the variable, ν has a uniform distribution. Suppose $\phi(\nu)$ is a bijective transformation function which has a one-to-one correspondence from the sample space of ν to the sample space of ϕ. Because of the same complete ignorance, the uniform distribution is assigned for ϕ

from the principle of insufficient reason. The transformation of the continuous probability density function (PDF) for ϕ provides PDF for ν, but the transformed PDF might not be a uniform distribution. Equations 4.2 through 4.5 show the relationship between the original and the transformed PDFs. The term in Equation 4.5, $\left|\frac{d\phi}{d\nu}\right|$ is called Jacobian of the transformation.

$$\int_{R_\nu} f'_\nu(\nu)d\nu = \int_{R_\phi} f'_\phi(\phi)d\phi \tag{4.2}$$

$$\int_{R_\nu} f'_\nu(\nu)d\nu = \int_{R_\phi} f'\left(\phi(\nu)^{-1}\right)\left|\frac{d\nu}{d\phi}\right|d\phi \tag{4.3}$$

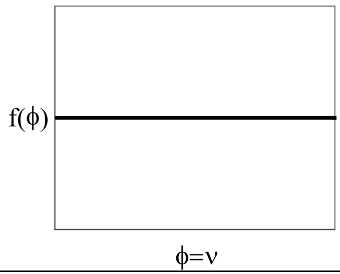
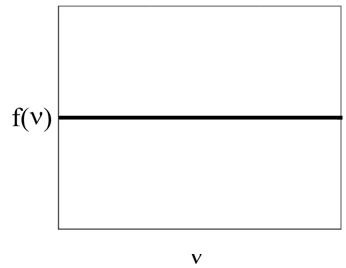
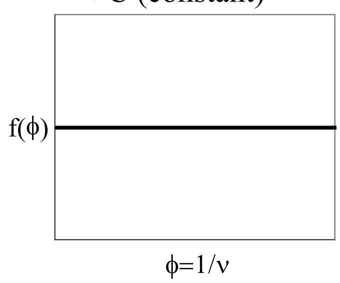
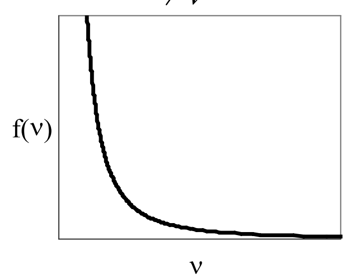
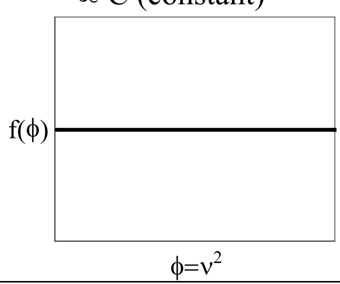
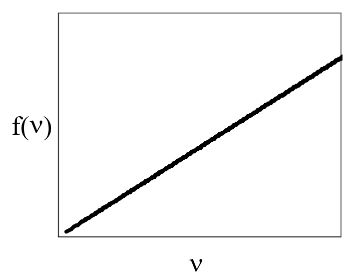
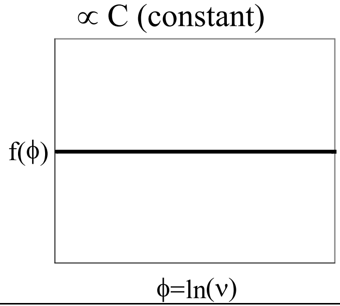
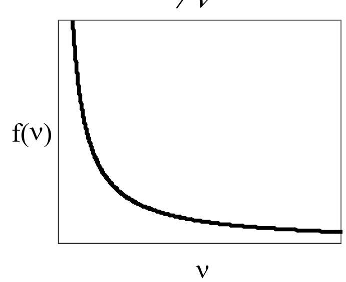
Therefore,

$$\int_{R_\phi} f'_\varphi(\phi)d\phi = \int_{R_\phi} f'\left(\phi(\nu)^{-1}\right)\left|\frac{d\nu}{d\phi}\right|d\phi \tag{4.4}$$

$$f'_\phi(\phi) = f'(\nu)\left|\frac{d\nu}{d\phi}\right| \text{ or } f'(\nu) = f'_\phi(\phi)\left|\frac{d\phi}{d\nu}\right| \tag{4.5}$$

Table 4.2 shows the various transformation functions and PDFs of the original and transformed variables. The transformation in parameter space may change the uniform distribution into the other form. This inconsistency questions on the transformation, that is, a sample space where the principle of insufficient reason should be applied to.

Table 4.2 Variations of uniform PDF by transformation of a parameter

Transformation Function	Jacobian	PDF $f_\phi(\phi)$	PDF $f_\nu(\nu)$
$\phi(\nu)=\nu$	$\left\lvert\frac{d\phi}{d\nu}\right\rvert=1$	$\propto$ C (constant) 	$\propto$ C (constant) 
$\phi(\nu)=\frac{1}{\nu}$	$\left\lvert\frac{d\phi}{d\nu}\right\rvert=\frac{1}{\nu^2}$	$\propto$ C (constant) 	$\propto 1/\nu^2$ 
$\phi(\nu)=\nu^2$	$\left\lvert\frac{d\phi}{d\nu}\right\rvert=2\nu$	$\propto$ C (constant) 	$\propto 2\nu$ 
$\phi(\nu)=\ln(\nu)$	$\left\lvert\frac{d\phi}{d\nu}\right\rvert=\frac{1}{\nu}$	$\propto$ C (constant) 	$\propto 1/\nu$ 

4.4 Violation of Decision Theory

4.4.1 Axioms in Decision Theory

Decision theory is a branch of study about choice. Decision theory includes descriptive and normative decision theories that provide how people make a decision and what the rational choice is, respectively. Rational behavior is expressed as a set of axioms in Luce and Raiffa (1957). They used axioms of decision theory in the case of complete ignorance to see whether each decision criterion fulfills them. Luce and Raiffa presented eleven axioms as follows. The nomenclature on axioms is from Milnor (1954).

Axiom 1. Ordering: Any decision problem can be resolved.

Axiom 2. Linearity: The choice set (optimal alternatives) for decision problems does not depend upon the choice of origin and unit of utility scale used to abstract problem.

Axiom 3. Symmetry: The choice set is invariant under the labeling of alternatives.

Axiom 4. Strong domination: If an alternative, A' belongs to a set of optimal alternatives and the other alternative, A'' has the same with or greater preference than A', A'' belongs to the set.

Axiom 5. Strong domination: If A' belongs to a set of optimal alternatives, A' is preferred to any other alternatives.

Axiom 6. Special row adjunction: Adding new alternatives, each of which is the same preference with some previous alternatives, has no effect on the preference in previous alternatives.

Axiom 7. Row adjunction: The addition of new acts does not transform an old, originally non-optimal act into an optimal one, and it can change an old, originally optimal act (the most preferred alternative) into a non-optimal one only if at least one of the new acts is optimal.

Axiom 8. Column linearity: Adding a constant utility to each consequence in a decision problem does not alter the optimal alternatives.

Axiom 9. Convexity: If A' and A'' are both optimal for a decision problem, a probability mixture of A' and A'' is also optimal.

Axiom 10. Symmetry: For any decision problems in complete ignorance, the optimal set should not depend upon the labeling of the state of nature.

Axiom 11. Column duplication: If a decision problem under uncertainty is modified by deleting a column which is equivalent to a probability mixture of other columns, then the optimal set (the most preferred alternative) is not altered.

A person may intuitively take some axioms such as Axiom 1 through 4 granted because the axioms are reflecting a general (and rational) human behavior. The human behavior is also the basis of decision criteria, for example, maximin, minmax, and maximum expected utility. Milnor (1954) summarized his work on axioms in decision theroy and decision criteria with Table 4.3. Table 4.3 illustrates Luce and Raiffa's point that Axiom 1 through Axiom 9 are compatible to maximum expected utility criterion while maximin, minimax, and Hurwicz are eliminated. Decision making under complete ignorance requires satisfying Axiom 10 and Axiom 11.

4.4.2 Violation of Axioms in Decision Theory

Luce and Raiffa (1957) discussed the state of complete ignorance on the basis of the axioms of decision criteria. They demonstrated that the principle of insufficient reason has drawbacks of providing an inconsistent probability distribution and an inconsistent optimal alternative for decision making problems under complete ignorance.

Table 4.4 shows the example similar to the example Luce and Raiffa used to illustrate the axiomatic violation with the principle. In this decision making problem, Decision Problem 1 (DP1), a decision maker has two decision alternatives, A_1 and A_2, and two possible states of nature under complete ignorance, S_1 and S_2. According to the principle, a decision maker assigns the same probability of ½ to both states of nature. These probabilities lead to an expected utility for each alternative, where A_1 is preferred to A_2, as shown in Table 4.5.

Table 4.3 Axiomatic compatibility of decision criteria (Milnor, 1954)

Axiom	Maximum Expected Utility	Maximin	Hurwicz	Minimax
1. Ordering	✓	✓	✓	✓
2. Linearity	✓	✓	✓	✓
3. Symmetry	✓	✓	✓	✓
4 & 5. Strong domination	✓	✓	✓	✓
6. Special row adjunction	✓	✓	✓	✓
7. Row adjunction	✓	✓	✓	
8. Column linearity	✓			✓
9. Convexity	✓	✓		✓
11. Column duplication		✓	✓	✓

The other decision problem, DP2, has the same two alternatives but a different number of states of nature. The additional four states of nature, S_3 through S_6 have the same consequences as with the state, S_2. Through the principle of insufficient reason, six states of nature have the same probability, 1/6. As shown in Table 4.6, A_2 is now preferred to A_1.

Table 4.4 Decision matrix for axiomatic approach example

		States of Nature	
		S_1	S_2
Decision Alternatives	A_1	11	0
	A_2	0	10

Table 4.5 Decision based on the principle of insufficient reason for Decision Problem 1 (DP1)

		States of Nature		Expected Utility
		S_1	S_2	
Decision Alternatives	$\mathbf{A_1}$	11	0	**5.5**
	A_2	0	10	5.0
Probabilities		$\frac{1}{2}$	$\frac{1}{2}$	

Table 4.6 Decision based on the principle of insufficient reason for Decision Problem 2 (DP2)

		States of Nature						Expected Utility
		S_1	S_2	S_3	S_4	S_5	S_6	
Decision Alternatives	A_1	11	0	0	0	0	0	1.8
	$\mathbf{A_2}$	0	10	10	10	10	10	**8.3**
Probabilities		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	

Under complete ignorance, DP1 and DP2 are identical because a decision maker does not know what the true state would be. If states of nature, S_2 through S_6 are lumped into one state S^*, probability that S^* will occur is equal to the sum of probabilities for S_2 through S_6, 5/6. For satisfying the axiom, S^* in DP2 should have the same probability with S_2 in DP1. However, $P(S^*)|_{DP2}=5/6$ and $P(S_2)|_{DP1}=½$. DP1 and DP2 are different from each other in principle of insufficient reason in that they have different probability distributions and different optimal decisions. This inconsistency indicates that the principle of insufficient reason violates Axiom 11.

The violation of Axiom 10 can be explained with the same decision example. Axiom 10 states that different labeling of states of nature, S_2 in DP1 and S^* (S_2 through S_6) in DP2 should not change the optimal set. However, the principle of insufficient reason produces different optimal decision, as shown in the example illustrated by Tables 4.5 and 4.6.

Axiom 10 alone does not mean that all states of nature are equally likely. Luce and Raiffa argued that the concept of equiprobability is a product of the combination of Axiom 10 and the previous seven Axioms (1 through 9 except 2 and 6). They unintentionally put one more constraint on the combination of eight Axioms. The constraint is the indifference in labeling of the states. The indifference plays an important role with the eight Axioms to produce equiprobable states of nature for decision making under complete ignorance. The indifference is not one of the axioms mentioned by them. It should be noted that Axiom 10 focuses on the optimal set, not on probabilities on states of nature, the indifference in states of nature, and/or the consequences in decision matrix.

4.5 EXTENSION OF PRINCIPLE OF INSUFFICIENT REASON

4.5.1 Entropy-Based Principles for Probability Assessment

Jaynes (1957) and Tribus (1969) made use of the principle of maximum entropy to assess the prior probabilities in Bayes' theorem. Their prior probabilities include information, which is associated in the form of average and/or variance when maximizing entropy. Jaynes (1957) suggested a principle of maximum entropy (PME) to establish the probability distribution on a random variable. The probability distribution from PME is unbiased because the maximization allows maximum uncertainty on the probability distribution subjected to a set of mathematical constraints, which is equivalent to previous knowledge. Jaynes argued that PME removes the arbitrariness in assigning probability distribution when information is provided as a mathematical form such as expected value or variance. Tribus (1969) showed various cases of probability distributions from different states of knowledge. The states of knowledge were formulated by mathematical equations such as the expected value, variance, and expected value of lognormal space. The equations work as constraints on the maximization of entropy. Table 4.7 shows probability distributions from the combinations of the formulated states of knowledge. The resultant PDFs are derived by taking limit on the number of discrete bins where the entropy is defined by.

A generalized version of the principle of maximum entropy is the principle of minimum relative entropy. Relative entropy of the probability density function (PDF), q(x), to the PDF, p(x), is defined with Equation 4.6, which is called the Kullback-Leibler relative entropy functional (Kullback, 1959). The PDF, p(x), represents previous knowledge and q(x) represents the state of knowledge associated with new information.

$$H(q, p) = \int q(x) \ln\left[\frac{q(x)}{p(x)}\right] dx \tag{4.6}$$

Table 4.7 Maximum entropy probability distributions

<table>
<tr><th>Constraints</th><th colspan="2">PDF from PME</th></tr>
<tr><td>$\sum p_i = 1$</td><td colspan="2">Uniform distribution</td></tr>
<tr><td>$\sum p_i = 1$
$\sum p_i x_i = \overline{x}$</td><td colspan="2">Exponential distribution</td></tr>
<tr><td rowspan="3">$\sum p_i = 1$
$\sum p_i x_i = \overline{x}$
$\sum p_i (x_i - \overline{x})^2 = \sigma_x^2$</td><td>$-\infty \le x \le +\infty$</td><td>Gaussian distribution</td></tr>
<tr><td>$0 \le x \le +\infty$</td><td rowspan="2">Truncated Gaussian distribution</td></tr>
<tr><td>Finite range of x</td></tr>
<tr><td>$\sum p_i = 1$
$\sum p_i x_i = \overline{x}$
$\sum p_i \ln(x_i) = \langle \ln(x) \rangle$</td><td colspan="2">Gamma distribution</td></tr>
<tr><td>$\sum p_i = 1$
$\sum p_i \ln(x_i) = \langle \ln(x) \rangle$
$\sum p_i \ln(1 - x_i) = \langle \ln(1 - x) \rangle$</td><td colspan="2">Beta distribution</td></tr>
</table>

The principle of minimum relative entropy means that the posterior probability distribution, q(x) should be as diffused as the prior probability, p(x), under the mathematical constraint from the new information such as expected values or variances.

If p(x) is a uniform distribution, the principle of minimum relative entropy is equivalent to the principle of maximum entropy.

The principle of maximum entropy and the principle of minimum relative entropy have been used for estimating prior probabilities in many fields. Baker (1990) applied them to uncertain variables of live loads on warehouse floors and friction angle of gravelly sand. Lind *et al*. (1991) studied the probability distribution for concrete strength on the basis of relative entropy. Woodbury and Ulrych (1993) performed Monte Carlo simulations with prior probability distribution for parameters affecting groundwater flow. In all cases, the prior probabilities were assigned by minimum relative entropy.

4.5.2 Invariance-Based Principle for Probability Assessment

Jeffreys (1946, 1961) and Box and Tiao (1973) introduced the concept of invariance to assess prior probabilities based on the principle of insufficient reason. Both of them suggested the way to have a single sample space where the principle of insufficient reason should be applied to. The basis of Jeffreys' rule is the discrepancy of a non-informative probability distribution in power transformation of sample space (Kass and Wasserman, 1994). Box and Tiao (1973) suggested the other method to establish a sample space for a non-informative prior probability distribution. Their fundamental concept to define the sample space is "data-translated likelihood", which means the invariance of the degree of uncertainty in any information. The notion indicates that various likelihood functions for any information have the same shape and different location in the sample space. Once the sample space is determined by the concept of data-translated likelihood, the uniform distribution is assigned based on the principle of insufficient reason.

Although the notions of Jeffreys and Box and Tiao, invariance, are essential to assess non-informative prior probabilities, the methods they suggested still require information. The information includes the probability models, such as Gaussian or logarithmic probability density function to qualify the invariance, Stieltjes discrepancy in Jeffreys' rule, and likelihood function in Box and Tiao's data-translated likelihood.

4.5.3 Limitations of the Entropy-Based and Invariance-Based Principles

Both of the groups, entropy-based camp and invariance-based camp, elaborated the methods to establish a sample space for non-informative prior probabilities. A unique sample space defined by them brings consistent non-informative prior probabilities. However, the non-informative prior probabilities are still subjected to the conclusion in Journel and Deutsch (1993), the maximum uncertainty in input variables may not guarantee the maximum uncertainty in responses. Therefore, the non-informative prior probabilities still based on the principle of maximum entropy applied to a sample space may not yield the decision alternatives under the maximum entropy. Furthermore, the both groups' methods are also based on some information, which may mislead the decision under the state of complete ignorance.

4.6 SUBJECTIVE PROBABILITIES

Based on three difficulties with the principle of insufficient reason, many well-known and reputed decision theorists, including Savage, Luce and Raiffa, Jaynes, etc, believe that theoretical difficulties with the principle are not relevant because there is always some subjective information to be used in assessing probabilities. They therefore defined the prior probabilities in applying Bayes' theorem (Equation 2.1) by including subjective information. The problem with this approach is that it is not defensible and does not provide a consistent starting point. Furthermore, in a complicated engineering problem where there are numerous states of nature, a tremendous amount of subjective information is required just to get started.

Chapter 5. Proposed Method: Decision-Based Non-Informative Prior Probabilities

5.1 Introduction

This chapter begins by proposing a definition of an unbiased starting point for decision making under complete ignorance. Based on this starting point, the basis of the decision-based method for non-informative prior probabilities will be developed, and a comparison will be made with the principle of insufficient reason. A detailed algorithm for implementation of the decision-based method in practice will be presented at the end of this chapter.

5.2 Unbiased Starting Point

The decision-based method is based on the fact that decision making under complete ignorance should be unbiased. In this section, the basis of the decision-based method and supporting argument for the concept will be provided. The basis of the decision-based method is about how the method is developed from the axioms of decision theory, and how the method provides an unbiased starting point for decision making. The supporting argument will show a connection between the decision-based method and the conventional concept of random choice.

5.2.1 Basis of Decision-Based Method

The basis of the sample space for a decision making under complete ignorance can be provided by the axioms in decision theory, Axiom 10 and Axiom 11. Those axioms characterize the state of complete ignorance and states that the labeling of the states of nature (the sample space) does not affect, and that adding repeated columns in decision matrix does not affect the non-informative prior probabilities. Those two axioms require the decision making problems with two decision alternatives in Figure 5.1 identical each other. When adding a new column (S_4 in Decision 2), the number of states of nature changes from 3 to 4, but there is no change in the number of preference outcomes. There are three possibilities of preference outcomes in Decision 1. In Decision 1, A_1 is preferred to A_2 at the state, S_1, A_2 is preferred to A_1 at S_2, and A_1 and A_2 has the same preference at S_3. In Decision 2, the state, S_4, provides the same preference outcome with the state, S_3, and the set of preference outcomes does not change.

		States of Nature		
		S_2	S_2	S_3
Decision	A_1	2	0	1
Alternatives	A_2	0	1	1

(a) Decision 1

		States of Nature			
		S_2	S_2	S_3	S_4
Decision	A_1	2	0	1	1
Alternatives	A_2	0	1	1	1

(b) Decision 2

Figure 5.1 Decision matrices for illustrating the basis of decision-based method

The sample space of preference outcome, which is implied by Axiom 10 and Axiom 11, provides another benefit: it consists of mutually exclusive and collectively exhaustive set of events. The preference outcomes between two decision alternatives can be classified into three possibilities: A_1 is preferred to A_2, A_2 is preferred to A_1, and A_1 and A_2 has the same preference:

$A_1 > A_2$

$A_1 < A_2$

$A_1 \sim A_2$

where ">" and "<" denotes dominance of one alternative to the others and "~" denotes the indifference between decision alternatives. These three preference outcome describe all possibilities in decision making and there is no overlap in the preference relationship.

The mutually exclusive and collectively exhaustive events provide an appropriate sample space to Bernoulli's principle of insufficient reason. The principle assigns the same probabilities to the sample space consists of preference outcomes and makes a decision making under complete ignorance unbiased. The resultant non-informative prior probabilities are based on the preference in all decision alternatives. This is why the proposed method is called a decision-based method.

A decision-based method can be extended to decision making with *n* decision alternatives. All possible preference outcomes for the decision making, which are mutually exclusive and collectively exhaustive, are as follows:

$$
\begin{aligned}
&A_1 > \\
&A_2 > \\
&\vdots \\
&A_n > \\
&A_1 \sim A_2 > \\
&A_1 \sim A_3 > \\
&\vdots \\
&A_{n-1} \sim A_n > \\
&\vdots \\
&A_1 \sim A_2 \sim A_3 > \\
&\vdots \\
&A_1 \sim A_2 \sim \ldots \sim A_n >
\end{aligned}
$$

Regardless of what the sample space is, it can be divided into these preference outcomes. The preference outcomes are discriminated by focusing on which decision alternative(s) is the most preferred. For example, the preference outcome, "A_1>" is distinguished from the preference outcome, "A_1~A_2>" because the first preference outcome represents the event that A_1 is the only best decision alternative, and the second one represents the event that both A_1 and A_2 are the most preferred to the others. The principle of insufficient reason is applied to the set of preference outcomes and all preference outcomes are equally likely. The probability is equal to the inverse of the total number of preference outcomes as in Equation 5.1.

$$p(\text{preference outcome}) = \frac{1}{{}_nC_1 + {}_nC_2 + \cdots + {}_nC_n} = \frac{1}{2^n - 1} \tag{5.1}$$

The resultant non-informative probabilities also make individual decision alternative equally likely. The event that A_i is the best decision includes all preference outcomes including A_i as one of the best alternatives, such as "A_1>", "A_1~A_2>" and "A_1~ A_2~ …~A_n>". In this manner, the probabilities that A_i is the best for all *i* (i=1, 2, …,*n*) are all equal and the probability is equal to $2^{(1-n)}$. The probabilities for the states of nature can be calculated by making all possible consequences for preferred alternative equally likely in a given preference outcome.

5.2.2 Complete Ignorance and Random Choice

Decision making under complete ignorance is called random choice. The principle of insufficient reason uses the concept of random choice by assigning the same probabilities to state of nature. It is an intuitive and understandable idea, but it may produce inconsistency and unreasonable probabilities, as shown in the previous chapter. Strictly speaking, the random choice in the application of the principle of insufficient reason is the choice with random states of nature. The decision-based method agrees with the idea that decision making under complete ignorance is random choice, but has a different interpretation of the concept of random choice. In this study, a random choice literally means a random choice. The randomness should go to a decision maker's choice, in other words, decision alternatives rather than to nature's choice (that is, states of nature). This interpretation is supported by Cohen and Jaffray (1980). They noted about the rational behavior under complete ignorance:

> "…since the decision maker has no special affection for any particular state of nature, the identities of the states of nature on which two given acts yield, respectively, such and such outcome have no effect on his preference between those acts."

Suppose a decision maker is playing a coin-flipping game. If a decision maker's guess is right, the reward would be 10 utilities and otherwise, -10 utilities. Based on the principle of insufficient reason, a decision maker assigns ½ to two states of nature: head (H) or tail (T) because there are two states of nature and because the randomness makes the probabilities, P(H) and P(T) equal to each other. The decision-based method also assigns the same probabilities to the two states of nature. However, the reason for assigning the same probabilities in the decision-based method is different from the one for the principle of insufficient reason. The decision-based method sees this decision making problem with newly decided two events with the same probabilities: an event that betting on H is preferred to betting on T and the other event for the opposite. The first case corresponds to the state, H and the second to the event, T. The probabilities on two states of nature are the same for the principle of insufficient reason and the decision-based method.

In fact, the ways used for the principle of insufficient reason implicates the decision-based method in this coin-flipping example. If a person thinks that the decision making under complete ignorance is based on the random states of nature, the states of nature would involved in uncertain factors in the game rather than the outcome of the game. For example, the uncertain factor might be the probability that the coin shows a head. Because there is no reason to believe that the coin is fair and no specification on the probability, it would be more reasonable for a conventional decision maker to assign the same probabilities to the sample space defined by P(H). However, what people do is different from the random state of nature. People set a sample space having the events, H and T rather than P(H) or P(T). The states of nature, H and T used in the principle of insufficient reason might be the sample space which a rational person under complete ignorance set unconsciously. The sample space with H and T is based on the preference in decision alternative and this is what the decision-based method does. This supports the statement that the decision-based method is valid because a rational person treats a decision making problem under complete ignorance as a random choice. The rational person may intuitively realize that the decision making under complete ignorance is a random choice and set up the sample space as above.

The fundamental idea of the decision-based method is that a starting point for decision making should be neutral to decision alternatives. The decision with these balanced alternatives is called an unbiased decision. This equality in the preference between decision alternatives turns decision making under complete ignorance into a random choice.

5.2.3 Random Choice and Sample Space

The new interpretation of the random choice can be implemented by manipulating states of nature to have a new sample space. In fact, the appropriate sample space has been a critical issue since Laplace. Laplace's equipossibility and equiprobability, Jeffreys' invariance, Box and Tiao's data translated likelihood are good examples for the efforts to make the rational non-informative probability distribution by defining a sample space. While the principle of insufficient reason focuses only on states of nature, the decision-based method extends the focus to a whole decision framework. The decision-based method has a modified interpretation of the uncertainty in decision making problem, as shown in Equation 5.2. The information in Equation 5.2 represents all available information given to a decision maker. $P(S_i|\text{Sample Space})$ becomes a non-informative prior probability distribution. The decision-based method makes use of decision frameworks to provide an appropriate sample space and non-informative probabilities. In the decision-based method, the sample space in Equation 5.2 is defined by preference outcomes from each state of nature. The preference outcomes are set to have balanced probabilities on the basis of random choice.

$$
\begin{aligned}
&P(S_i \mid \text{Information}, \text{Sample Space}) \\
&\qquad = \frac{P(\text{Information} \mid S_i, \text{Sample Space})\, P(S_i \mid \text{Sample Space})}{P(\text{Information} \mid \text{Sample Space})}
\end{aligned}
\tag{5.2}
$$

The idea of manipulating the sample space by preference outcomes comes from scenario-based planning. In scenario-based planning, a probability distribution is discarded and an uncertainty is reduced to "a few scenarios whose differences make a difference to decision-makers" (Schwartz, 1991) or to "a small set of fundamentally different paths into the future" (Lempert *et al.*, 2006). Based on this concept of scenario-based approach, Lempert *et al.* developed a robust decision making (RDM) method for deep uncertainty, which is equivalent to complete ignorance. The method reduces a large number of states of nature into two new states: a state where a certain alternative is preferred and the other state where the alternative is not preferred. However, RDM may not provide consistent result in the two states when the decision is made under three or more decision alternatives. In addition, as mentioned in their study, RDM does not determine the best decision alternative; it is intended to help a decision maker to compare the behavior of each alternative.

In the viewpoint of a decision maker, the scenario-based approach provides new states of nature, which is extensively simplified from the original sample space and the new states of nature also requires probabilities. Although the scenario-based approach is not complete in that the method does not have a criterion on identifying a single optimal alternative, the idea is one of the motivations of this research. The first lesson from the scenario-based approach is that the states of nature can be transformed into new states of nature. The second lesson is that the new states of nature are selected by their impact or importance on decision alternatives (in other words, the consequences in decision matrix).

In the previous decision example in Section 4.2 (Figure 4.1 and Table 4.1), States 1 and 2 have the preference in decision alternatives, Alternative A>Alternative B (Alternative A is preferred to Alternative B), and State 3 has Alternative A<Alternative B. The original sample space (States 1, 2, and 3) turns into the new sample space of two preference outcomes, as shown in Figure 5.2. For non-biased decision under complete ignorance, P(Alternative A>Alternative B) should be equal to P(Alternative A<Alternative B). These equal probabilities mean that the principle of insufficient reason or its modern version of the principle of maximum entropy (PME) is applied to

the new sample space: Event 1, (Alternative A>Alternative B) and Event 2, (Alternative A<Alternative B). The probability for each state in the new sample space is ½, as shown in Figure 5.3.

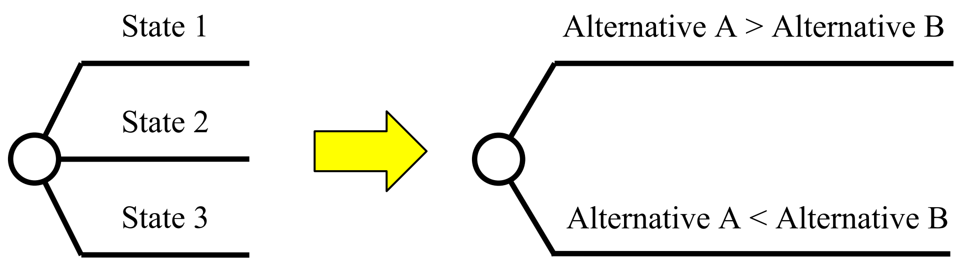


Figure 5.2 Concept of the decision-based method: modification in sample space

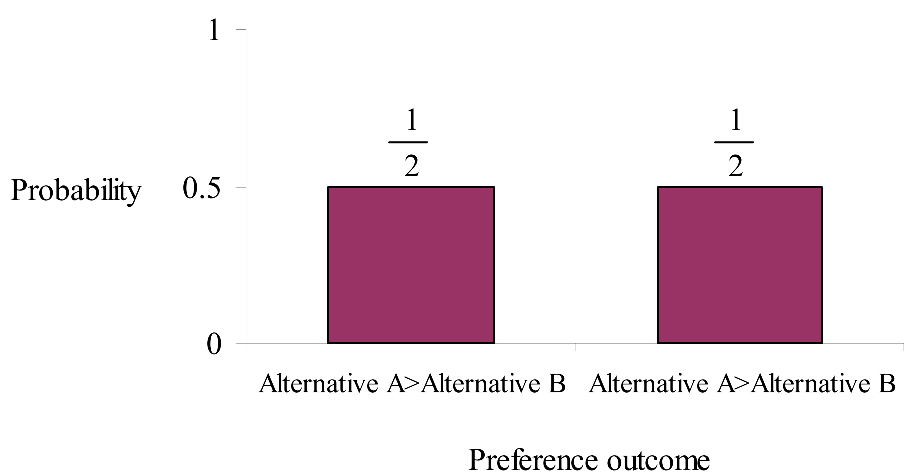


Figure 5.3 Unbiased starting point for decision making

For decision analysis, a probability distribution on the original sample space is required. The PME can be used again for this conversion of the non-biased probability distribution into the original states of nature. The previous application of the principle of insufficient reason provides a set of constraints on the sample space transformation. The constraints are built from the relationship between the original states of nature and preference outcome space. Each preference outcome has corresponding state of nature. Event 1 (Alternative A>Alternative B) corresponds to States 1 and 2 and Event 2 (Alternative A<Alternative B) to State 3. In mathematical expression, P(Event 1)=$P(S_1)+P(S_2)$ and P(Event 2)=$P(S_3)$. Therefore, the optimization by PME is the maximizing the function, H, defined in Equation 5.3 subjected to the constraints, Equations 5.4 and 5.5.

$$H = -\{P(S_1)\ln(P(S_1)) + P(S_2)\ln(P(S_2)) + P(S_3)\ln(P(S_3))\} \tag{5.3}$$

$$P(S_1) + P(S_2) = \frac{1}{2} \tag{5.4}$$

$$P(S_3) = \frac{1}{2} \tag{5.5}$$

Applying the principle of insufficient reason (or the principle of maximum entropy) with these constraints yield the same probabilities for S_1 and S_2. The resultant non-informative prior probability distribution for the simple example is shown in Figure 5.4. Although the non-informative probabilities in Figure 5.4 are not uniform, the PMF indicates the same preference in two decision alternatives. This means that the PMF in Figures 5.3 and 5.4 represents an unbiased starting point for the simple decision making example.

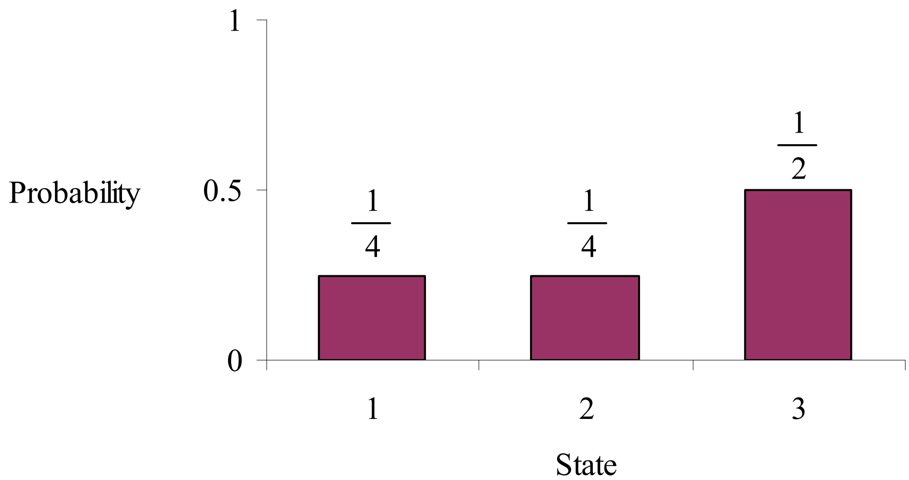


Figure 5.4 Non-informative prior probability distribution for simple decision example

5.3 ALGORITHM FOR DECISION-BASED METHOD

In the previous section, the decision-based method based on the unbiased starting point was shown with a simple example. The detailed algorithm for the decision-based method will be explained in this section with a larger decision matrix. The algorithm for the decision-based method is implemented with a Visual Basic code in the form of a module. The form provides a flexible way to be coupled with decision making applications. The algorithm code is attached in Appendix B.

Suppose we have a decision making problem with no information, as shown in Table 5.1 and Figure 5.5. The first task of the decision-based method is to build a set of preference outcomes from each state of nature in the decision matrix. There are four possible preference outcomes:

1. A_1>: A_1 is the only most preferred decision alternative
2. A_2>: A_2 is the only most preferred decision alternative

3. A_3>: A_3 is the only most preferred decision alternative

4. $A_1 \sim A_2$>: Both A_1 and A_2 are the most preferred decision alternative

The first preference outcome is from the state, S_1, the second preference outcome from the state, S_2, the third from the state, S_7, and the fourth from the states, S_3 through S_6. The second task is to apply Bernoulli's principle of insufficient reason to the preference outcomes. Each of four preference outcomes takes the probability of ¼, as shown in Figure 5.6, and this represents the unbiased starting point for decision making. The third task is to convert the probabilities assigned to preference outcomes to probabilities on the original states of nature, S_1 through S_7. Each of the states of nature, S_1, S_2, and S_7 is the only state corresponding to each preference outcome. These relationships between preference outcomes and states of nature are expressed in Equations, 5.6 through 5.8.

$$P(S_1)=P(A_1>)=1/4 \tag{5.6}$$

$$P(S_2)=P(A_2>)=1/4 \tag{5.7}$$

$$P(S_7)=P(A_3>)=1/4 \tag{5.8}$$

The four states of nature, S_3 through S_6 have the same preference outcome, $A_1 \sim A_2$>.

$$P(S_3)+P(S_4)+P(S_5)+P(S_6)=P(A_1 \sim A_2>)=1/4 \tag{5.9}$$

The rest four states of nature, S_3 through S_6 are discriminated with the consequence of the most preferred decision alternative. The state, S_6 has a utility of 8 for A_2 while the states, S_3, S_4, and S_5 have 10. These two subsets should have the same probabilities of 1/8.

$$P(S_3)+P(S_4)+P(S_5)=P(S_6)=1/8 \tag{5.10}$$

From the notion of the complete ignorance expressed with Axiom 10 and Axiom 11, the sum of non-informative probabilities for S_3 and S_4 should be equal to the probability for S_5.

$$P(S_3)+P(S_4)=P(S_5)=1/16 \tag{5.11}$$

The principle of insufficient reason is again applied to have the unknown probabilities, $P(S_3)$ and $P(S_4)$ with the mathematical constraint given in Equation 5.11. Therefore, the probabilities for S_3 and S_4 are equal to 1/32. The resultant non-informative prior probabilities for the states of nature are shown in Figure 5.7.

Table 5.1 Decision matrix for a decision making example with three decision alternatives

		States of Nature							Expected Utility
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	
Decision Alternatives	A_1	11	10	0	0	0	0	0	5.25
	A_2	0	10	10	10	10	8	1	5
	A_3	9	9	7	7	5	2	2	6
Probabilities		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	

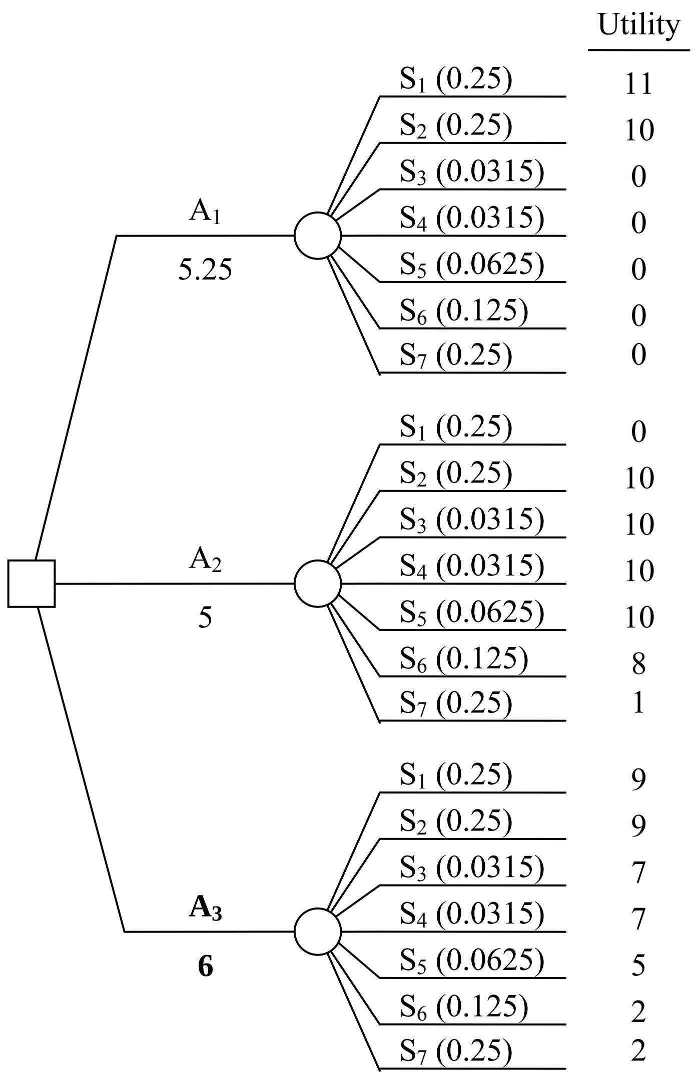


Figure 5.5 Decision tree for a decision making example with three decision alternatives

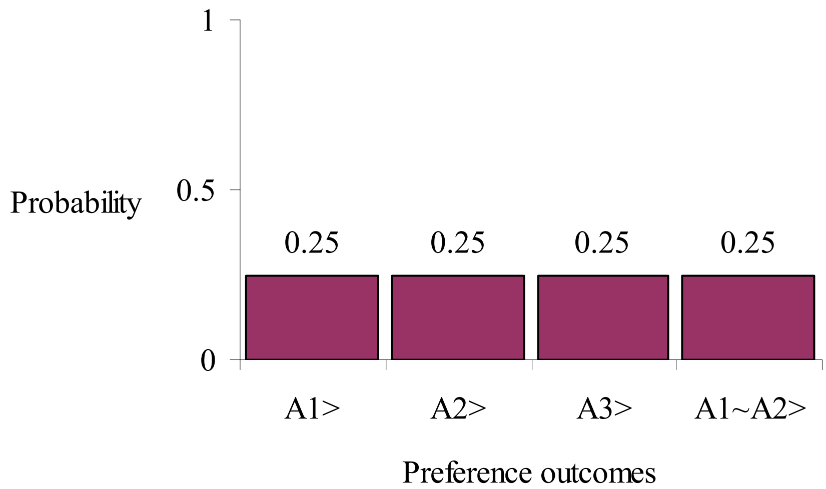


Figure 5.6 Non-informative prior probabilities assigned to preference outcomes

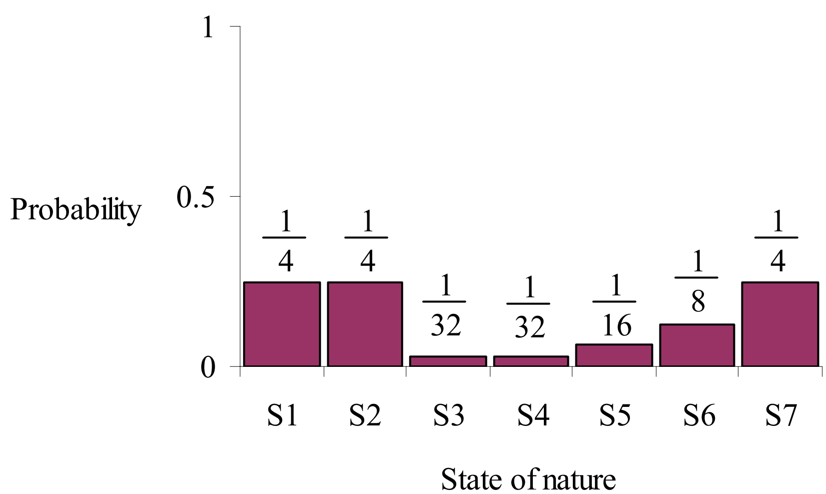


Figure 5.7 Non-informative prior probabilities assigned to states of nature

Chapter 6. Evaluation and of Decision-Based Method for Non-Informative Prior Probabilities

6.1 INTRODUCTION

A detailed discussion on the decision-based method will be given. The objective of this chapter is to evaluate the merits of the proposed decision-based method for non-informative prior probabilities (decision-based priors). The decision-based method will be evaluated according to the following three questions: 1) does the decision-based method produce rational and reasonable results?, 2) can the decision-based method be applied consistently and practically so that a decision maker always gets the same starting point for the same problems?, and 3) does the decision-based method satisfy axioms of decision theory? In addition, the practical implications of this proposed approach will be explored.

6.2 EVALUATION OF DECISION-BASED PRIORS

The principle of insufficient reason for establishing non-informative prior probability distribution has three practical difficulties. It provides an unbiased starting point for decision making, it does not lead to consistent results, and it violates reasonable axioms of decision theory.

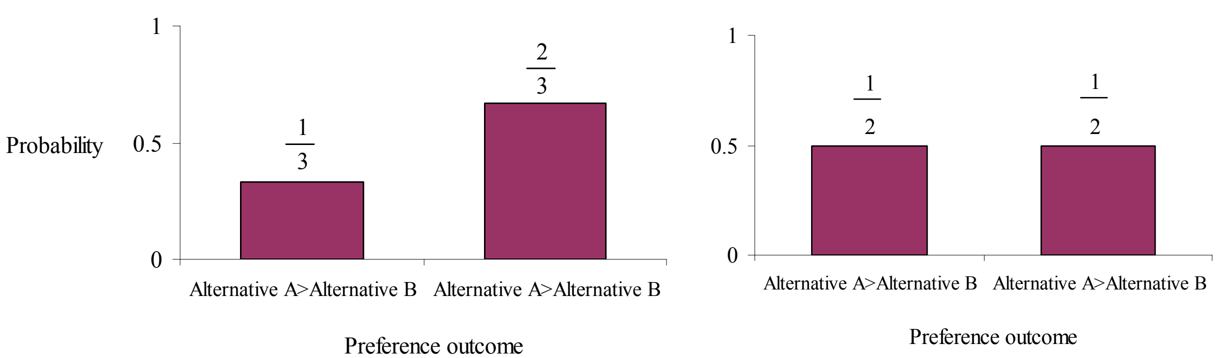
6.2.1 Unbiasedness

The underlying principle of decision-based priors is to make all possible outcomes of decision preference equally likely. Therefore, it is designed specifically to produce unbiasedness in the decision alternatives. The previous example in Table 4.1, Figure 4.1, and Figure 5.3 can illustrate the unbiasedness, as shown in Table 6.1 and

Figure 6.1. While the principle of insufficient reason assigns different probabilities to each preference outcomes (Figure 6.1a), the decision-based method results in the equally likely preference outcomes (Figure 6.1b).

Table 6.1 Decision matrix for illustrating unbiasedness of the decision-based method

		States of Nature		
		State 1	State 2	State 3
Decision Alternative	Alternative A	0	1	2
	Alternative B	3	2	0
Probabilities	Principle of insufficient reason	⅓	⅓	⅓
	Decision -based method	¼	¼	½
Preference Outcome		Alternative A< Alternative B	Alternative A< Alternative B	Alternative A> Alternative B



(a) Principle of insufficient reason

(b) Decision-based method

Figure 6.1 Unbiasedness in decision-based method

6.2.2 Consistency

The method to establish decision-based prior probabilities provides a consistent set of probabilities for the states of nature because it relies solely on how the states affect the decision consequences and not how the states themselves are defined. Therefore, for the same decision making problem (that is, the same set of alternatives and possible consequences for each alternative), consistent probabilities will be obtained for the possible consequences regardless of how the associated states are labeled or defined.

The consistency in decision-based priors can be illustrated with the heterogeneous porous media example in Section 4.3.1. A decision framework is assigned to the heterogeneous porous media example, as shown in Table 6.2. In this decision matrix, two preference outcomes are possible, as shown in Figure 6.1. The decision-based method assigns probabilities to two different sample spaces on states of nature, as shown in Figure 6.2. The non-informative probabilities given in forms of bivariate k (Figure 6.2 (a) left) can be transformed into the sample space of k_{eff} (Figure 6.2 (a) right). Therefore, whether k_1 and k_2 or k_{eff} are used to define the states of nature, the same result is obtained

There are tow possible preference outcomes, $A_1>A_2$ for the state, S_1 and $A_1<A_2$ for the states, S_2 through S_4. The decision-based method assigns the same probabilities to those two preference outcome:

$$P(A_1>A_2)= ½$$

$$P(A_1<A_2)= ½$$

Because the state, S_1 is the only state which has the preference outcome, $A_1>A_2$ and the states, S_2 through S_4 have the preference outcome, $A_1<A_2$,

$$P(S_1)= P(A_1>A_2)= ½$$

$$P(S_2)+P(S_3)+P(S_4)= P(A_1<A_2)= ½$$

Based on the axioms in decision theory regarding a decision making under complete ignorance, the states, S_2 and S_3 (the repeated columns), could be treated as a divided states from the lumped state, S^*, and they are equally likely.

$$S^*=\{S_2, S_3\}$$

$$P(S_2)=P(S_3)$$

The principle of insufficient reason is applied to the states of nature, S^* and S_4, with the same preference outcome. The application of the principle is subjected to mathematical constraint on the summation of probabilities equal to ½. The results are:

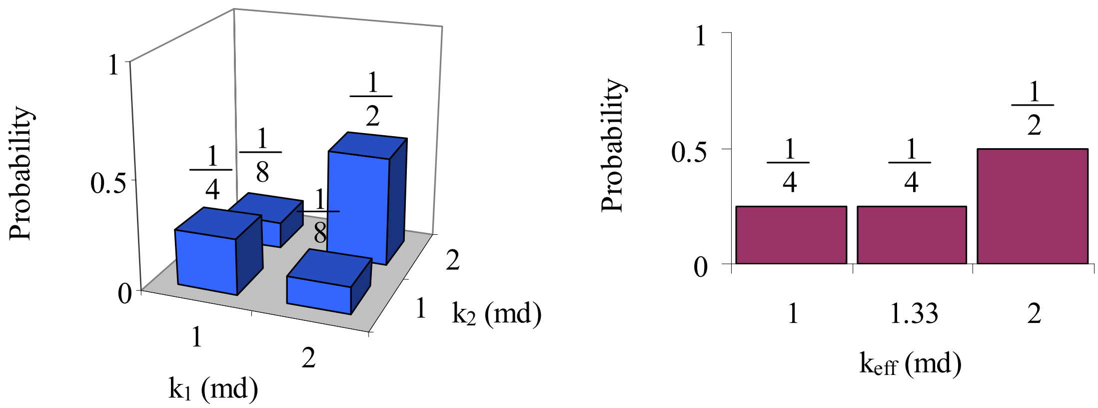
$$P(S^*)=P(S_4)= 1/4$$

$$P(S_2)=P(S_3)= 1/8$$

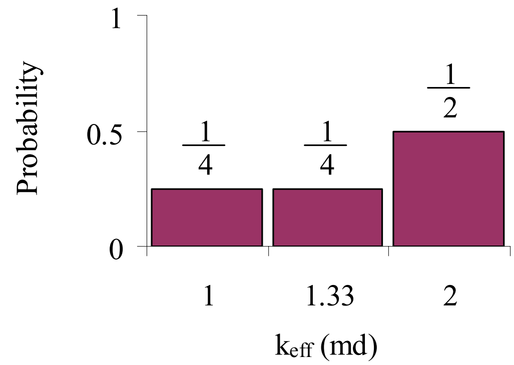
These decision-based priors are shown in Figure 6.2.

Table 6.2 Decision matrix for the heterogeneous porous media example

		State of Nature			
		S_1	S_2	S_3	S_4
		$[k_1,k_2]=[1,1]$	$[k_1,k_2]=[1,2]$	$[k_1,k_2]=[2,1]$	$[k_1,k_2]=[2,2]$
		$k_{eff}=1$	$k_{eff}=1.33$	$k_{eff}=1.33$	$k_{eff}=2$
Decision Alternatives	A_1	10	5	5	2
	A_2	2	7	7	7
Preference Outcome		$A_1>A_2$	$A_1<A_2$	$A_1<A_2$	$A_1<A_2$
Probability		½	⅛	⅛	¼



(a) Uniform distribution on input variables, k_1 and k_2



(b) Uniform distribution on a response variable, k_{eff}

Figure 6.2 Consistent non-informative prior probabilities for heterogeneous porous media example by the decision-based method

6.2.3 Satisfaction of Axioms in Decision Theory

The method for decision-based priors satisfies two axioms in decision theory for complete ignorance proposed by Luce and Raiffa (1957):

> Axiom 10. Symmetry: For any decision problems [*sic*] in complete ignorance, the optimal set should not depend upon the labeling of the state of nature.

Axiom 11. Column duplication: If a decision problem under uncertainty is modified by deleting a column which is equivalent to a probability mixture of other columns, then the optimal set (the most preferred alternative) is not altered.

The same examples used to show axiomatic violations of the principle of insufficient reason will be used to compare the decision-based method and the principle of insufficient reason.

The method of decision-based priors satisfied Axiom 11 because duplicate columns provide the same consequences, which are treated as equally likely possibilities. Therefore, the contribution of uncertainty in this consequence remains the same no matter how many different states of nature produce it.

To illustrate Axiom 11, decision making problems, DP1 and DP2 in Tables 4.5 and 4.6, respectively have the same preferred decision. The resultant probability distributions from the decision-based method can be obtained with the preference outcomes shown in Figures 6.3 and 6.4. Tables 6.3 and 6.4 are non-informative PMFs for both decision making problems. For DP1, both the principle of insufficient reason and the method for decision-based priors have the same probability distribution, but for DP2, the probability distribution is different from each other, as shown in Table 6.5.

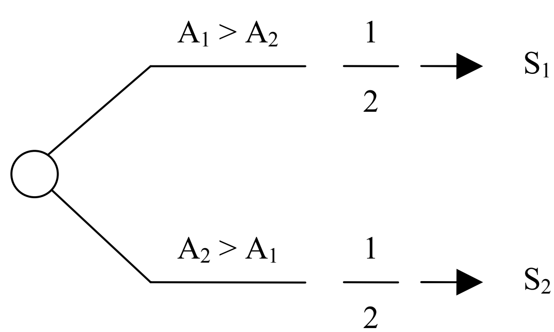


Figure 6.3 Preference outcomes for Decision Problem 1 (DP1)

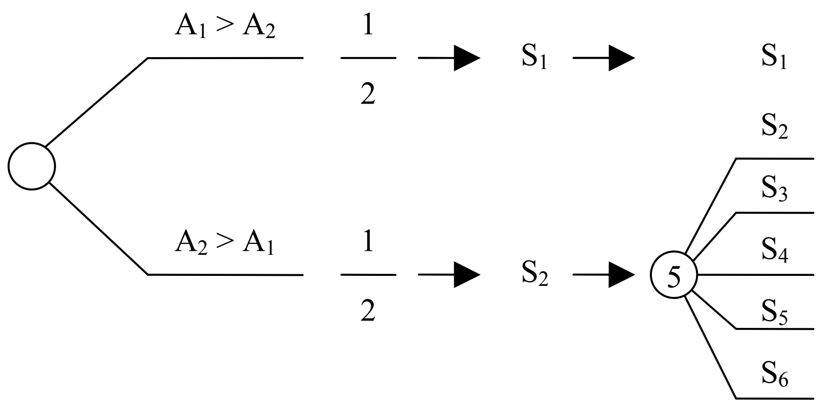


Figure 6.4 Preference outcomes for Decision Problem 2 (DP2)

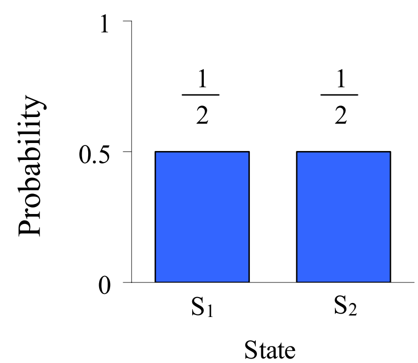
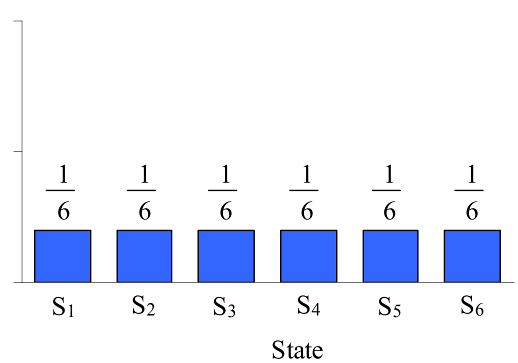
Table 6.3 Decision based on probabilities from the decision-based method for Decision Problem 1 (DP1)

		States of Nature		Expected
		S_1	S_2	Utility
Decision Alternatives	**A_1**	11	0	**5.5**
	A_2	0	10	5.0
Probabilities		$\frac{1}{2}$	$\frac{1}{2}$	

Table 6.4 Decision based on probabilities from the decision-based method for Decision Problem 2 (DP2)

		States of Nature						Expected
		S_1	S_2	S_3	S_4	S_5	S_6	Utility
Decision Alternatives	**A_1**	11	0	0	0	0	0	**5.5**
	A_2	0	10	10	10	10	10	5.0
Probabilities		$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	

Table 6.5 Probability distributions under complete ignorance for DP1 and DP2 by the principle of insufficient reason and the decision-based method

<table>
<tr><th></th><th>Decision Problem 1</th><th>Decision Problem 2</th></tr>
<tr><td>Principle of Insufficient Reason</td><td>
</td><td>
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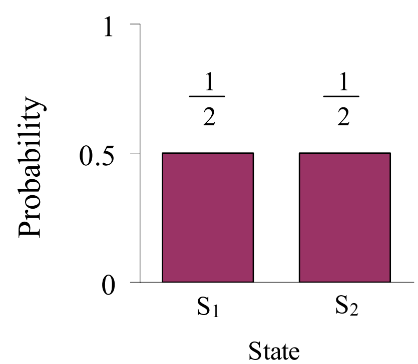
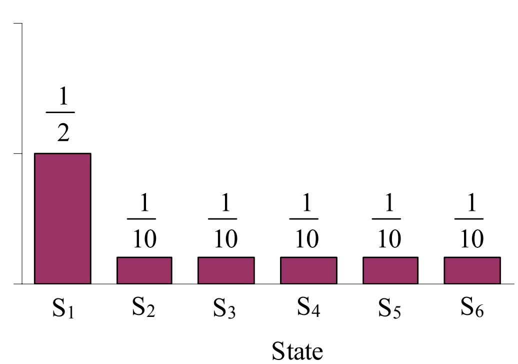
<tr><td>Decision-Based Method</td><td>
</td><td>
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</table>

While the principle of insufficient reason provides two different optimal decision alternatives on two decision making problems (A_1 for DP1 and A_2 for DP2), the decision-based method provides the same optimal decision alternative for both decision alternatives in DP1 and DP2 (Table 6.6). The expected utility for A_1 is 5.5 and that for A_2 is 5.0 for DP1 and DP2. The decision-based method therefore satisfies Luce and Raiffa's concept on complete ignorance in decision making: S_2 in DP1 is equivalent to S_2 through S_6 in DP2 for a decision maker.

Axiom 10 requires consistency in the prior probabilities, regardless of how the states in the sample space are defined. As derives in Section 6.2.2, the method of decision-based priors provides consistency and satisfies Axiom 10.

The satisfaction of Axiom 10 can be illustrated with heterogeneous porous media example shown in Figure 4.3. The uncertain variable in the example can be expressed with individual permeabilities, k_1 and k_2 or with effective permeability, k_{eff}. These two ways to label the uncertain variable caused inconsistency in non-informative probabilities, as shown in Figure 4.4, and the change in the optimal decision, as shown in Table 6.7 for the principle of insufficient reason. However, the decision-based method provides the same non-informative probabilities, expected utilities, and the optimal decision.

Table 6.6 Comparison of axiomatic satisfaction (Axiom 11) between the principle of insufficient reason and decision-based methods

	The Principle of Insufficient Reason	Decision-Based Method
Decision Problem 1	$E(A_1)=5.5$ $E(A_2)=5.0$ A_1 is optimal	$E(A_1)=5.5$ $E(A_2)=5.0$ A_2 is optimal
	$P(S_1)=0.5$ $P(S_2)=0.5$	$P(S_1)=0.5$ $P(S_2)=0.5$
Decision Problem 2	$E(A_1)=1.8$ $E(A_2)=8.3$ A_2 is optimal	$E(A_1)=5.5$ $E(A_2)=5.0$ A_1 is optimal
	$P(S_1)=0.16$ $P(S_2)+P(S_3)+P(S_4)+P(S_5)+P(S_6)=0.84$	$P(S_1)=0.5$ $P(S_2)+P(S_3)+P(S_4)+P(S_5)+P(S_6)=0.5$
DP1 and DP2 have	- Difference decisions - Incompatible probabilities $P(S_2) \neq P(S_2)+P(S_3)+P(S_4)+P(S_5)+P(S_6)$	- Same decisions - Compatible probabilities $P(S_2)=P(S_2)+P(S_3)+P(S_4)+P(S_5)+P(S_6)$
DP1 and DP2 are	Not identical	Identical

Table 6.7 Decision analysis results for both sample spaces of heterogeneous porous media example

Sample Space	The Principle of Insufficient Reason	Decision-Based Method
$[k_1,k_2]$, 4 Bins	Expected utility for A_1=5.5 Expected utility for A_2=5.75 Optimal decision: A_2	Expected utility for A_1=4.75 Expected utility for A_2=5.75 Optimal decision: A_2
$[k_{eff}]$, 3 Bins	Expected utility for A_1=5.67 Expected utility for A_2=5.33 Optimal decision: A_1	Expected utility for A_1=4.75 Expected utility for A_2=5.75 Optimal decision: A_2

6.3 Practical Implications of the Decision-Based Priors

6.3.1 Scope of the Method for Decision-Based Priors

The method for decision-based priors is based on the context of a decision. The method is therefore not applicable to a probability assessment problem without a decision framework. For example, the proposed approach is not applicable for calculating the probability of failure of an offshore structure system, the probability that the oil price increases, the probability that a 6.0-magnitude earthquake strikes the nuclear facility, and so on. However, the point of assessing probability is ultimately to support decision making, so that limitation is not significant. In fact, it is telling that probabilities need a decision framework to have any meaning.

6.3.2 Change in Non-Informative Probabilities with Different Decision Frameworks

The method for decision-based priors requires a decision matrix for a decision making problem as an input, and accordingly may be influenced by a change in decision. Non-informative prior probabilities for different decision making problems may be different from each other, even in the case of decision making problems with the same states of nature. The different non-informative probabilities are due to different decision alternatives or to different consequences.

Suppose we have two different decision making problems, as shown in Tables 6.8 and 6.9. Those two decision making problems have the same set of states of nature - S_1, S_2, and S_3 - but different decision alternatives. The proposed method to assign decision-based prior probability, shown in Figures 6.5 and 6.6, yields two different non-informative prior probabilities, as shown in Figure 6.7. The only difference between DP3 and DP4 is one additional decision alternative in DP4. Adding A_3 changes the structure of preference outcomes, the non-informative probabilities, expected utilities, and the optimal decisions. DP3 has A_1 as an optimal decision because the expected utilities for A_1 and A_2 are 5.5 and 5, respectively. DP4 has A_2 as an optimal decision because expected utilities for A_1, A_2, and A_3 are 3.67, 6.67, and 6, respectively. This does not mean that the decision-based method violates Axiom 7 in decision theory (see Section 4.4.1) because by changing decision we have changed the sample space for the possible states of nature. The decision criterion based on maximum expected utility satisfies Axiom 7 when the same sample space is used in both decision making problems.

Table 6.8 Decision matrix for Decision Problem 3 (DP3)

		States of Nature		
		S_1	S_2	S_3
Decision Alternatives	A_1	11	0	0
	A_2	0	10	10

Table 6.9 Decision matrix for Decision Problem 4 (DP4)

		States of Nature		
		S_1	S_2	S_3
Decision Alternatives	A_1	11	0	0
	A_2	0	10	10
	A_3	3	12	3

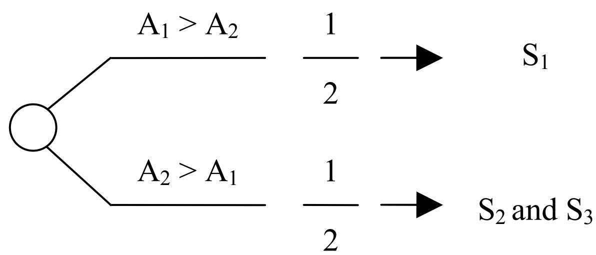


Figure 6.5 Preference outcomes for Decision Problem 3 (DP3)

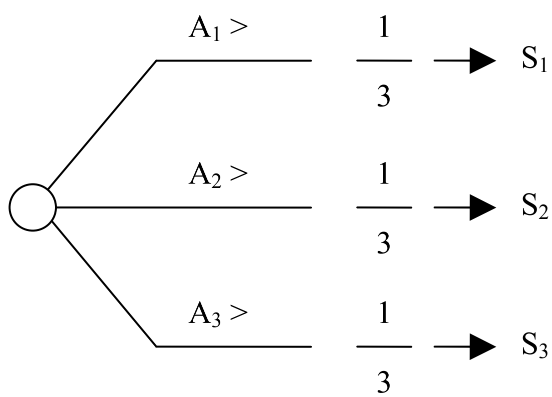
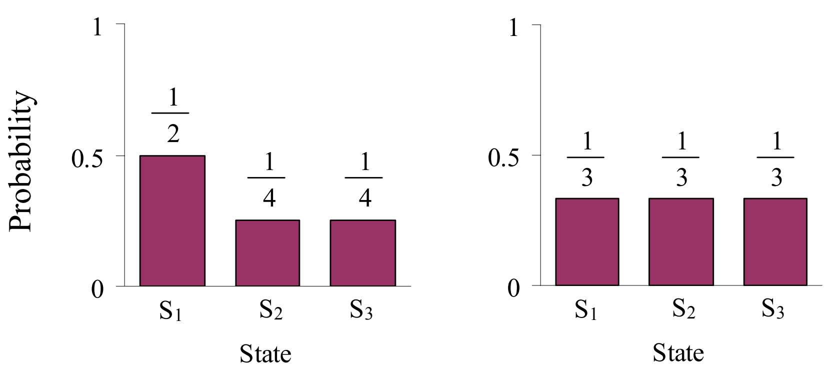


Figure 6.6 Preference outcomes for Decision Problem 4 (DP4)



(a) Decision Problem 3 (b) Decision Problem 4

Figure 6.7 Decision-based non-informative prior probabilities for DP3 and DP4

6.3.3 Non-Informative Probabilities and Decision with Information

Any additional information can be incorporated in a decision making problem through Bayes' theorem, shown in Section 2.3. The available information is expressed as a likelihood function that relates the probabilities of having obtained the information for each possible state of nature. The non-informative prior probabilities from the method of decision-based priors and the likelihood function are associated through Bayes' theorem, and the resultant posterior probabilities go to a decision making problem with information, as shown in Figure 6.8.

The method of decision-based priors is advantageous, specifically in practical decision making with information, in that the method can reduce the subjectivity. In practical decision making, the information is usually vague because of heterogeneity in population, sparsity of data, and error in measurement. Because it is difficult to formulate decision making problems, it is common to make mathematical assumptions (e.g. Gaussianity or mathematically convenient probability density function) in order to establish a starting point for decision making. The method of decision-based priors provides an unbiased and consistent starting point for decision making without making subjective assumptions.

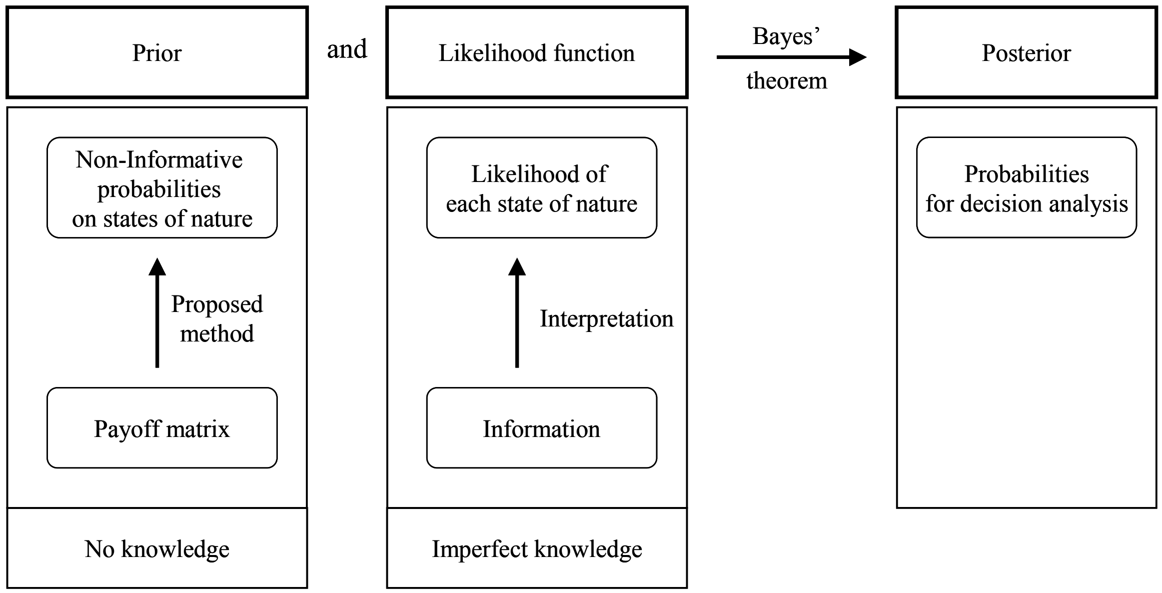


Figure 6.8 Schematic diagram for incorporation of information through a Bayesian framework

6.3.4 Non-Informative Probability Densities for Continuous Sample Space

The decision-based method can be extended to decision-making problems with a continuous sample space for the state of nature. The algorithm for the decision-based method is basically the same for both continuous and discrete sample spaces. Suppose we have a decision making problem with complete ignorance characterized by utility functions, as shown in Figure 6.9. The continuous random variable representing the state of nature ranges between 0 and 6, and there are two decision alternatives called Alternative 1 and Alternative 2. For $0<x<4.34$, Alternative 1 is preferred to Alternative 2, and for $4.34<x<6$, Alternative 2 is preferred to Alternative 1. In a discrete sample space, the discrete sample space of Δx represented by each bin is considered to have the same utility. In other words, the intervals are discretized by utility (y-axis) rather than the state of nature, x (x-axis), as shown in Figures 6.10 and 6.11. The discretized x, Δx, might not be equally discretized as demonstrated by different Δx_i when the utility function is non-linear.

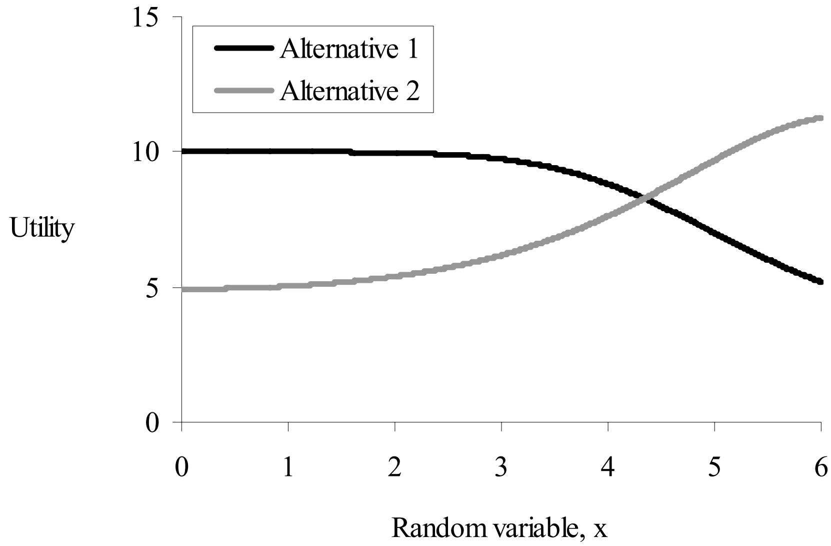


Figure 6.9 Utility functions for two decision alternatives with a sample space defined by a random variable, x

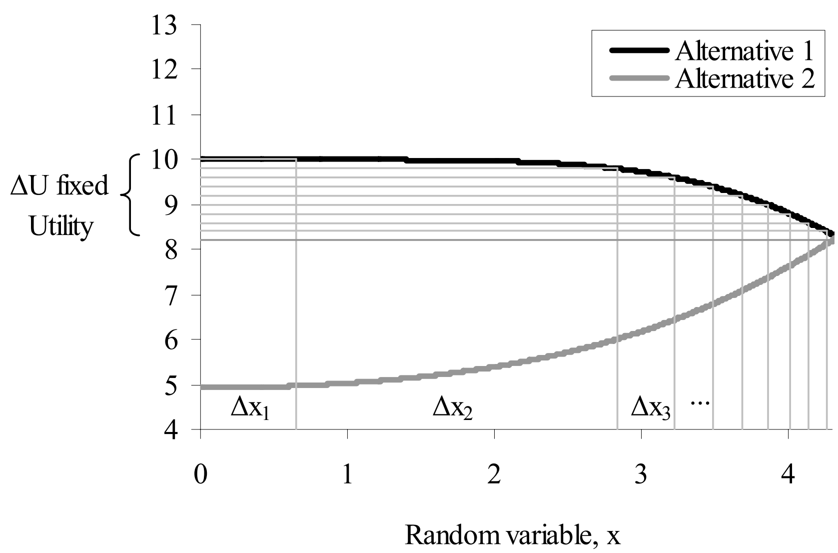


Figure 6.10 Discretized continuous sample space where Alternative 1 is preferred to Alternative 2

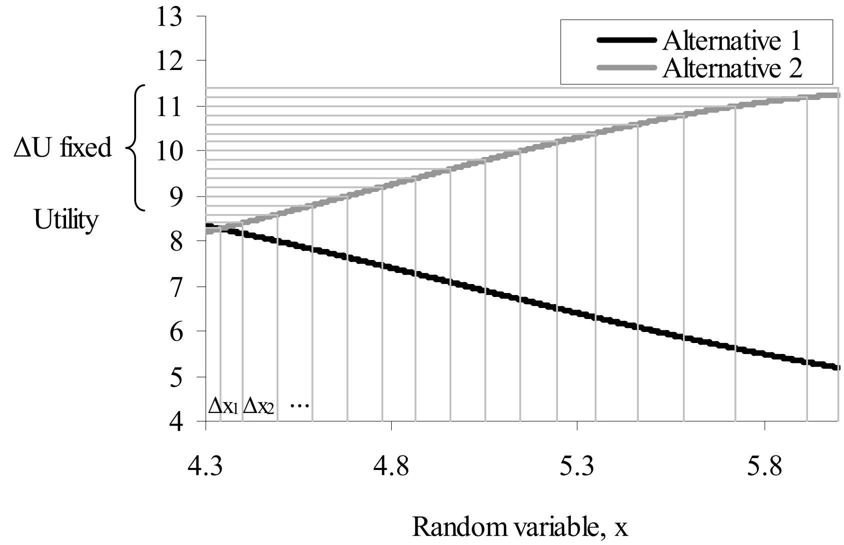


Figure 6.11 Discretized continuous sample space where Alternative 2 is preferred to Alternative 1

The physical basis of this method of discretization is that a decision maker could not distinguish two states of nature, x and x+Δx, giving close consequences such as u(x) and u(x+Δx). If Δu goes to zero, this interpretation produces a non-informative probability density function proportional to |du/dx|.

This non-informative probability density function yields a consistent result for any transformation of the random variable, x. Consider a different labeling for the states of nature, y, where y=g(x). The non-informative prior probability density function for x and y within each possible decision preference would be found as follows.*

* Note: This derivation assumes that u is a monotonic function of x and y. It could readily be extended to more complicated utility functions.

$$f_X(x) \propto \left|\frac{du}{dx}\right| \tag{6.1}$$

$$f_Y(y) \propto \left|\frac{du}{dy}\right| \tag{6.2}$$

Hence, the non-informative prior density function, $f_X(x)dx$ and $f_Y(y)dy$ are proportional to one another whether x or y is used to define the states of nature. If x is used to establish a non-informative prior PDF and then transformed to be in terms of y,

$$f_Y(y) \propto f_X(x)\left|\frac{dx}{dy}\right| \tag{6.3}$$

Conversely, if y is used directly to establish a non-informative prior PDF,

$$\begin{aligned} f_Y(y) &\propto \left|\frac{du}{dy}\right| \\ &\propto \left|\frac{du/dx}{dy/dx}\right| \propto f_X(x)\left|\frac{dx}{dy}\right| \end{aligned} \tag{6.4}$$

Therefore, the same PDF for y is obtained either way and consistency (or invariance) is satisfied.

For the algorithm of the decision-based method, the non-informative PDF for the uncertain variable x is shown in Figure 6.12. A given decision matrix (Figure 6.9) provides two possible preference outcomes: Alternative 1 is preferred to Alternative 2, and Alternative 2 is preferred to Alternative 1. The same probabilities are assigned to those preference outcomes:

P(Alternative 1>Alternative 2)=½

P(Alternative 1<Alternative 2)=½

Because each preference outcome corresponds to the range of random variable, x, the assigned probabilities can be expressed with the range:

P(0<x<4.34)=½

P(4.34<x<6)=½

For the range, 0<x<4.34, where Alternative 1 is preferred to Alternative 2, the non-informative PDF is established by Equation 6.1 with the utility function for Alternative 1. For the other range, 4.34<x<6, the utility function in Equation 6.1 is set to be the utility function for Alternative 2, which is preferred to Alternative 1 in the range. The resultant non-informative PDF is shown in Figure 6.12.

There is a drop at x=4.34 because the preference outcomes on the left and right are different from each other. Each of P(0<X<4.34) and P(4.34<X<6) has the probability of ½. The logarithmic transformation function, f(x)=ln(X) yields the non-informative prior PDF, as shown in Figure 6.13.

If a decision maker starts with a different sample space, y, the decision matrix is illustrated with Figure 6.14. The non-informative prior PDF for y is shown in Figure 6.15. Because y is set as a function of x, y=ln(x), the PDF in Figure 6.15 is identical to Figure 6.13. This example illustrates the consistency in non-informative PDFs obtained from the proposed method.

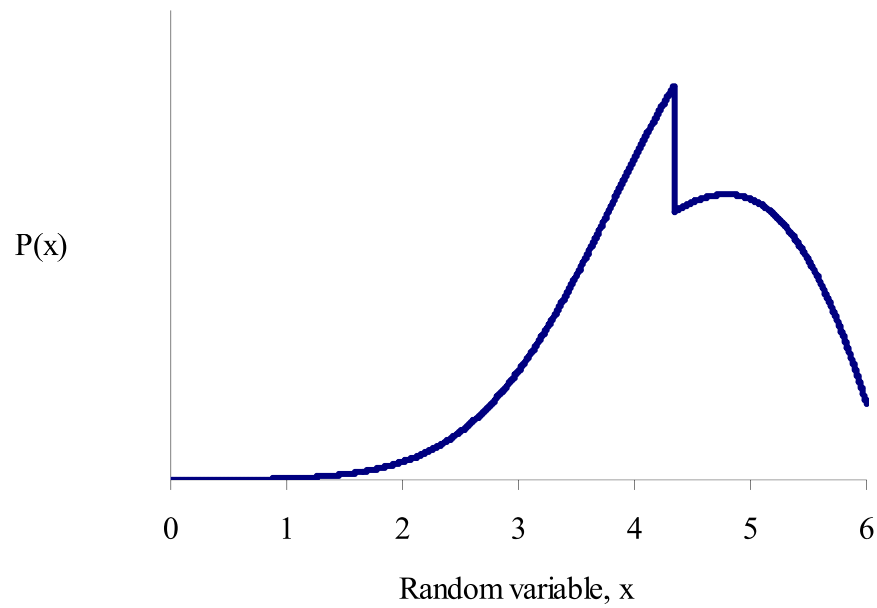


Figure 6.12 Non-informative prior probability density function for x

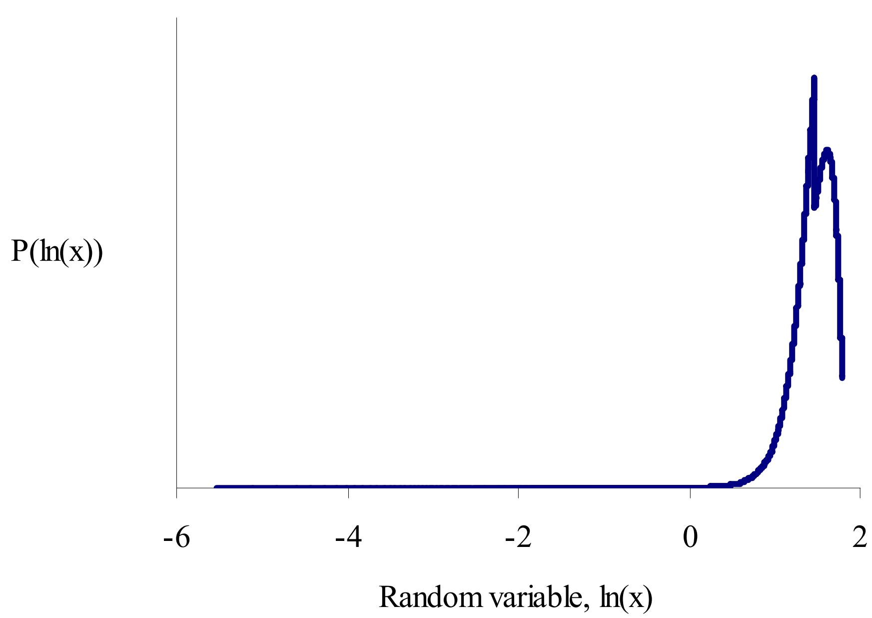


Figure 6.13 Non-informative prior probability dendsity function for the transformed sample space, ln(x)

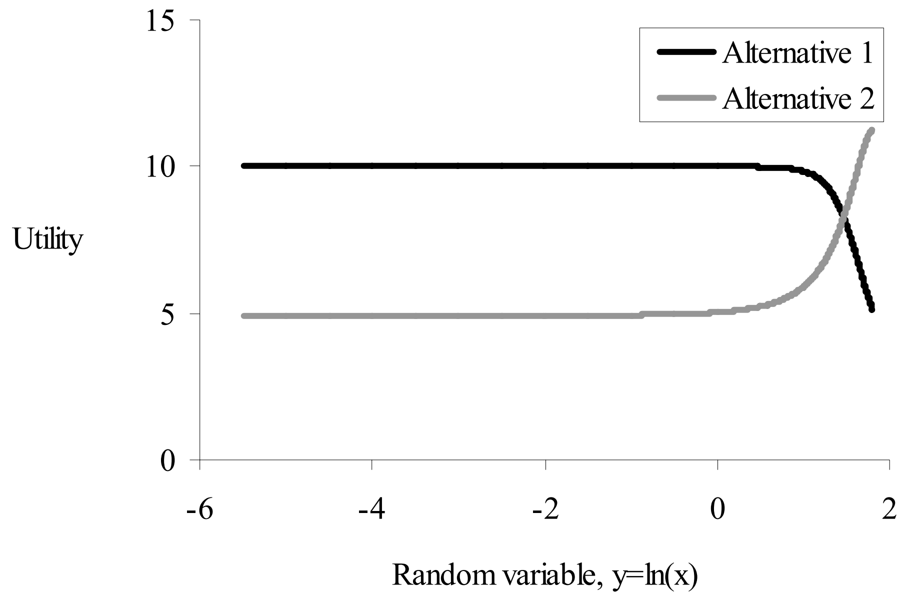


Figure 6.14 Utility functions for two decision alternatives with a sample space defined by a random variable, y

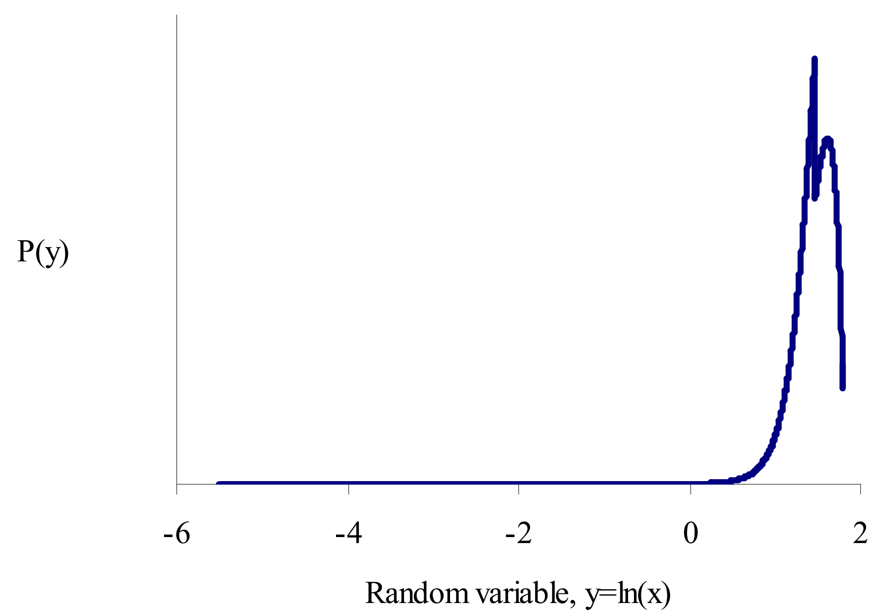


Figure 6.15 Non-informative prior probability density function for y

6.3.5 Design of the Sample Space for Non-Informative Probabilities

A sample space used in decision analysis is also an input for decision-based non-informative prior probabilities, as shown in Equation 5.2. A decision maker may have to make a decision about the sample space - for example, the range (lower and/or upper bound) of a random variable.

The proposed decision-based method provides two requirements for the sample space: the sample space should include all possible preference outcomes and should be extended to the range or bins with constant consequence. Those two requirements are related with the algorithm of the proposed method based on decision outcomes that include preference outcomes and decision consequences.

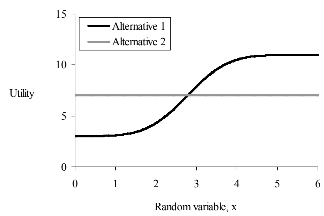
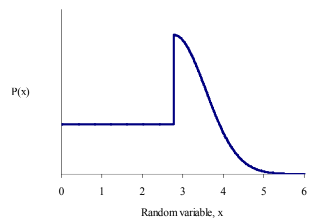
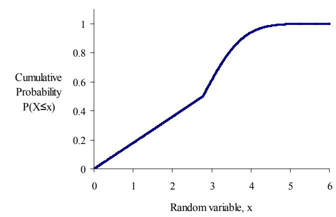
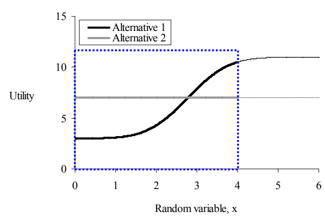
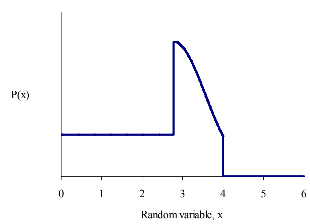
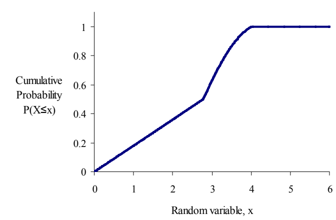
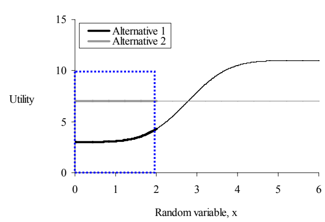
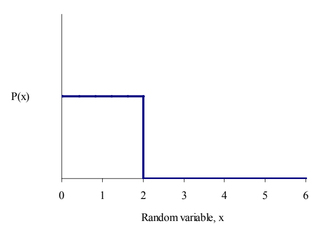
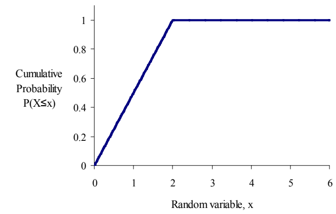
Suppose a decision maker has a continuous random variable, x, two decision alternatives, A_1 and A_2, and consequences, as shown in the first row of Table 6.10. There are two possible preference outcomes, $A_1>A_2$ and $A_1<A_2$, and the range of x covers $du/dx \approx 0$ (where $x \approx 5$). The range of $0 \leq x \leq 6$ satisfies the two requirements. The second case of the range, $0 \leq x \leq 4$, has two preference outcomes, but du/dx does not reach zero ($du/dx \approx 1.1$). This range may cause inconsistency when the random variable is transformed to the other. The third case of the range, $0 \leq x \leq 2$, does not satisfy both requirements. The range covers only one preference outcome. The derivative, du/dx at x=2, is not approaching zero. Because of the missing preference outcome, the decision-based non-informative probability density function for the third case is very different from the first and the second cases.

The two requirements do not mean that a decision maker must have the infinite range of random variables. A decision maker can use any range of random variables which satisfy the two requirements. For the given utility functions in Table 6.10, the consequences of A_1 and A_2 are constant for x larger than 5. The decision-based non-informative prior density functions for any range that has an upper bound larger than 5 will consistently provide the same expected utilities of A_1 and A_2, and accordingly the same optimal decisions.

In practical decision making, the consequence reaches the maximum value before the random variable, x, does not reach the extreme. For example, a decision maker may be concerned with extreme permeabilities - for example, 1,000 or 10,000 (md). In this case, the productivity index for the reservoir and the peak production rate are large, so that the pipe line capacity may constrain the production rate as the constant. The two extreme values of permeability, 1,000 and 10,000 (md), may not affect consequences in a decision matrix, because the production rate will be constant during a production life of interest. In other words, two different states of nature do not make any difference in decision outcomes including preference in decision alternatives and decision consequences. In this case, a decision-based method allows removing one of two states of nature without any change in expected utilities for decision alternatives. Therefore, a decision maker's concern is unnecessary.

If a decision maker is sure about the range of random variables, the range can be used as a sample space for decision making, regardless of the two requirements from the proposed decision-based method. The decision-based method will still provide an unbiased starting point for decision making with the given set of sample space. For example, a reservoir engineer wants to define a sample space for porosity of the reservoir. The sample space does not need to cover the porosity value greater than unity by its definition. However, if a decision maker is considering a range of porosity between 20 and 30 (%), care should be taken based on the discussion in the previous paragraph.

Table 6.10 The effect of the range of a random variable on decision-based non-informative prior probability

Range of x	Utility Function	Probability Density Function	Cumulative Density Function
[0,6]			
[0,4]			
[0,2]			

Chapter 7. Application of Proposed Method to Engineering Decision Making Problems

7.1 INTRODUCTION

The objective of this chapter is to apply the decision-based method to engineering decision making problems. The decision examples will show how the method works for practical decision making. The practical implications of the decision-based method addressed in the previous chapter will be explained with the decision examples. The decision examples focus on hydrocarbon exploration and production, which involves a large set of decision alternatives and significant uncertainty in the states of nature.

7.2 PRODUCTION EXAMPLE

Cullick *et al.* (2003) tested their optimization algorithm for decision making with a case history. The objective of the decision case was maximizing the value of oil recovery from three units of a reservoir by determining the optimal production schedule. In this section, the example will be adapted and used to study non-informative probabilities for decision making analysis.

7.2.1 Objectives

This example is designed to make two points. The first is to show the difference between non-informative prior probabilities from the principle of insufficient reason and the decision-based methods. The difference will be visualized by comparing joint and marginal probability distribution of uncertain variables. The second purpose is to show that the decision-based method is practically feasible in decisions with a large number of decision alternatives and states of nature.

7.2.2 Decision Description

A company makes an investment on a new oil field that has two possible reservoir units (Figure 7.1). A decision is made by comparing a variety of plans for production and finding the optimal plan that maximizes the expected value of net profit. The investment plan includes the number of production wells for each unit and when they are installed over 10 years of production life. The following assumptions are made for decision alternatives:

1. The maximum number of wells for each unit is set 4.
2. Either or both units should be in production at year 1.
3. A unit can be installed and abandoned at any time in a 1 year interval.
4. The oil production from a unit is controlled in a gathering center, not in individual wells in the unit.
5. Once the valve in a gathering center is closed, it is not allowed to open later, but crossflow between two reservoir units can occur.

The assumptions build 12,241 decision alternatives which consist of 12,240 Go options and 1 No go (Abandon) option. Each alternative can be described with six decision variables: the number of wells, and years for the beginning and ending oil production for Units 1 and 2. For example, Alternative 3412 means that the oil production is from 2 production wells in Unit 1 for 10 years, and from 4 production wells in Unit 2 for 8 years (through years 3 and 10). The description on Alternative 3412 can be illustrated with Figure 7.2.

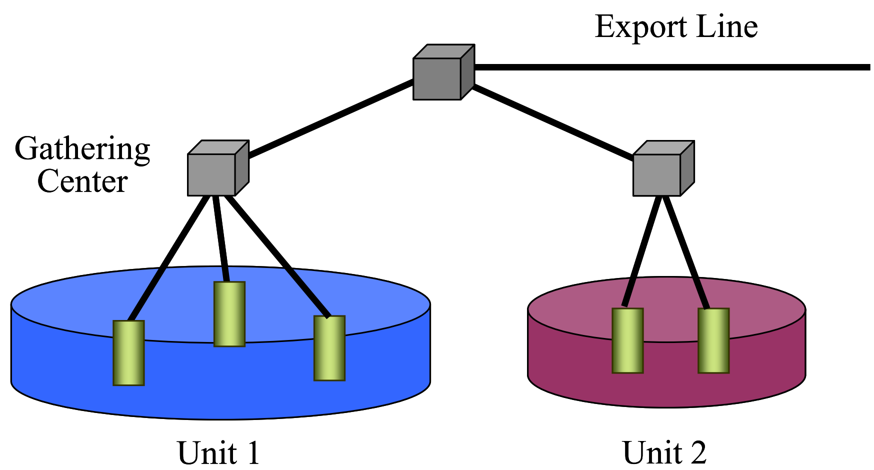


Figure 7.1 Production system for the production example

	Year	1	2	3	4	5	6	7	8	9	10
Alternative 3412	Unit 1	2 wells									
	Unit 2			4 wells							

Figure 7.2 Description on Alternative 3412 for the production example

There are many physical parameters affecting oil production associated with this decision case, including the reservoir drainage area, net pay, temperature, reservoir discontinuity, oil viscosity, permeability, skin factor, transmissibility between reservoir units, and pipeline flow capacity. Because the decision is made based on monetary value, the economic parameters - oil price, costs for well installation, facility construction, maintenance, and inflation rate - are also important. It is assumed that the uncertain variables include porosity, permeability, oil price, and cost for each unit. The ranges of the variables are given in Table 7.1. The rest of the parameters are constant, as shown in Table 7.2.

Table 7.1 Range of variables that are uncertain in the production example

Variable		Range
Porosity, ϕ	(%)	1-40
Permeability, k	(md)	0.1-1,000
Oil price	($/bbl)	30-50
Well cost	($/well)	5,000,000-10,000,000
Facility cost	($/Unit)	12,000,000-20,000,000

Table 7.2 Assumed physical parameters for the production example

Parameter		Value
Drainage area, A	(acres)	5000 for Unit 1 2500 for Unit 2
Net pay thickness, h	(ft)	300
Total compressibility, c_t	(psi^{-1})	0.00005
Oil viscosity, μ	(cp)	0.8
Initial reservoir pressure, P_{ini}	(psi)	2,500
Designated wellbore pressure, P_{wf}	(psi)	2,000
Radius of wellbore, r_w	(ft)	0.5
Oil formation volume factor, B_o	(RB/STB)	1.17
Shape factor, C_A	(-)	30.88
Skin factor, s	(-)	0
Maximum production rate, q_{Lim}	(bbl/day)	20,000

7.2.3 Decision Framework

The decision alternatives and associated uncertainty in this example can be illustrated with the decision tree in Figure 7.3. The uncertainty is modeled as discrete scenarios with states of nature that represent a set of possible combinations of eight uncertain variables, as shown in Figure 7.4. The uncertain variables include petrophysical parameters (porosity and permeability) and economic parameters (oil price and cost). The uncertainty model consists of 50,625 (=225^2) states of nature.

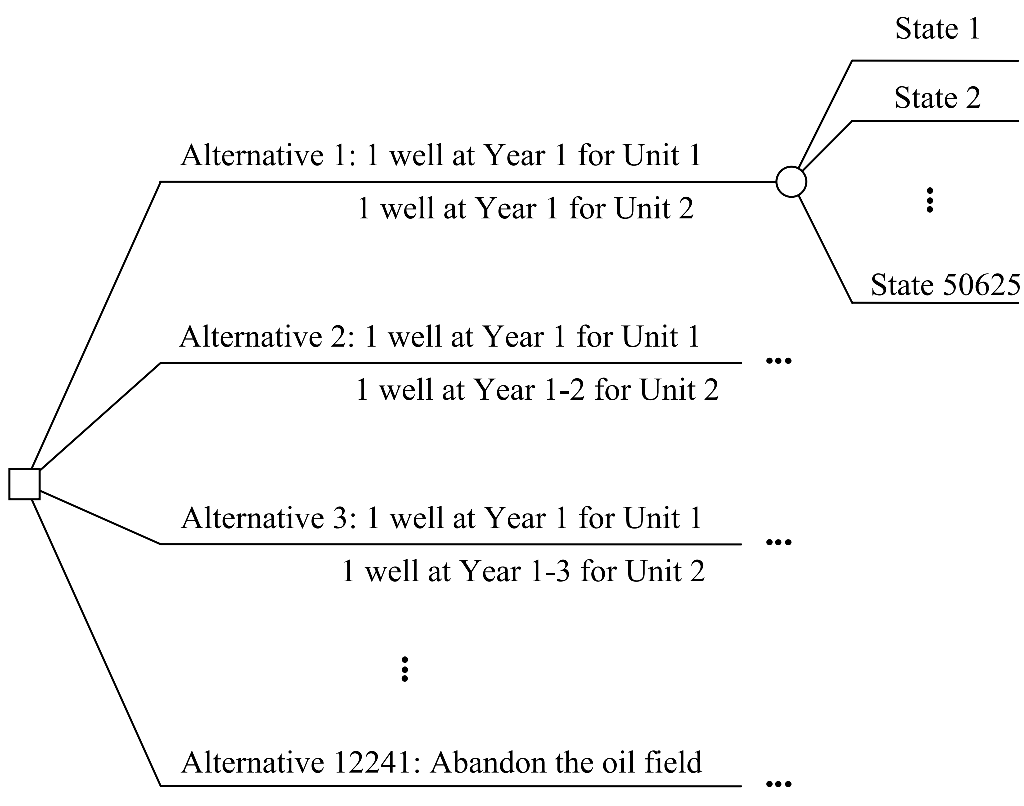


Figure 7.3 Decision tree for the production example

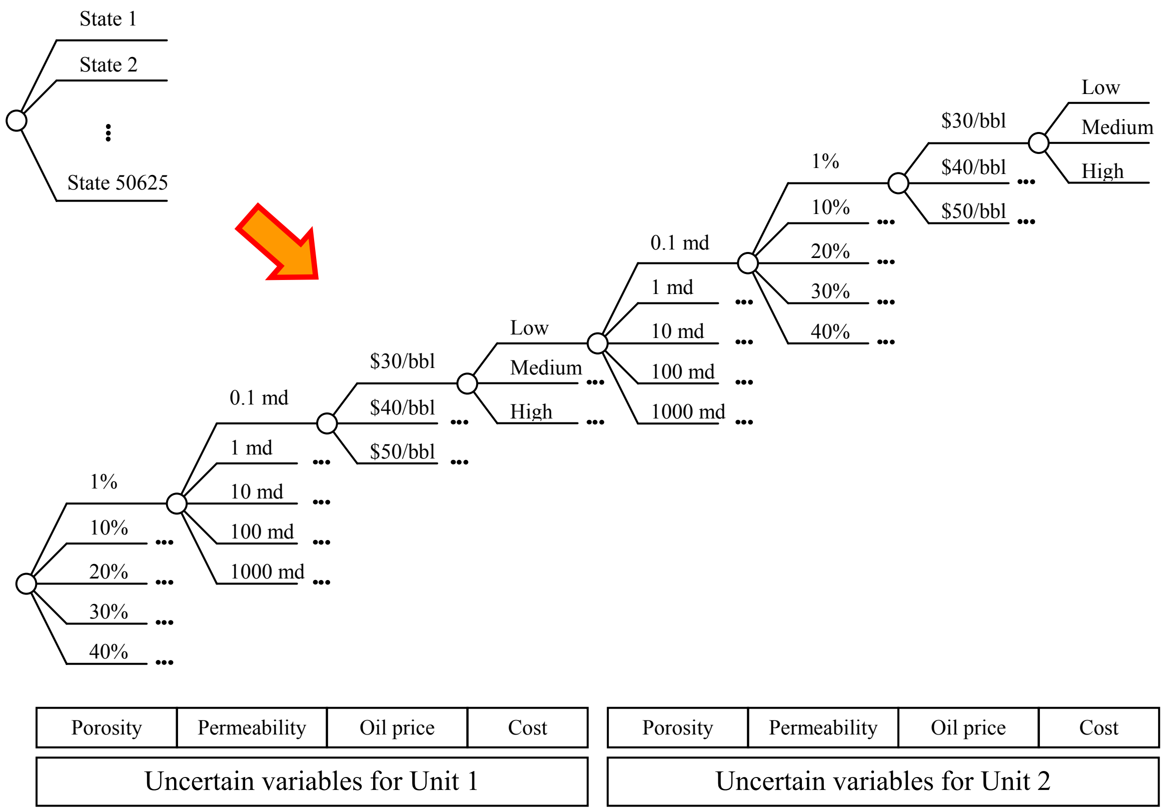


Figure 7.4 Event tree for possible states of nature in the production example

The consequence of each pair of a decision alternative and a state of nature is quantified by the net present value (NPV) of the net profit made by the pair. To link the NPV and the pair, the modified tank model (in Section 7.3.2) is used and economic analysis is conducted. The modified tank model helps obtain the time history of oil production as it relates to physical variables. The time histories are used as input to cash flow analysis (CFA), as shown in Figure 7.5. Oil price, cost, and discount factor are involved in the CFA to estimate a discounted NPV of the net profit for a given pair of decision alternative and state of nature. The process for estimating NPV is illustrated in Figure 7.6. The NPV was the measure of the consequence used by Cullick *et al.* (2003).

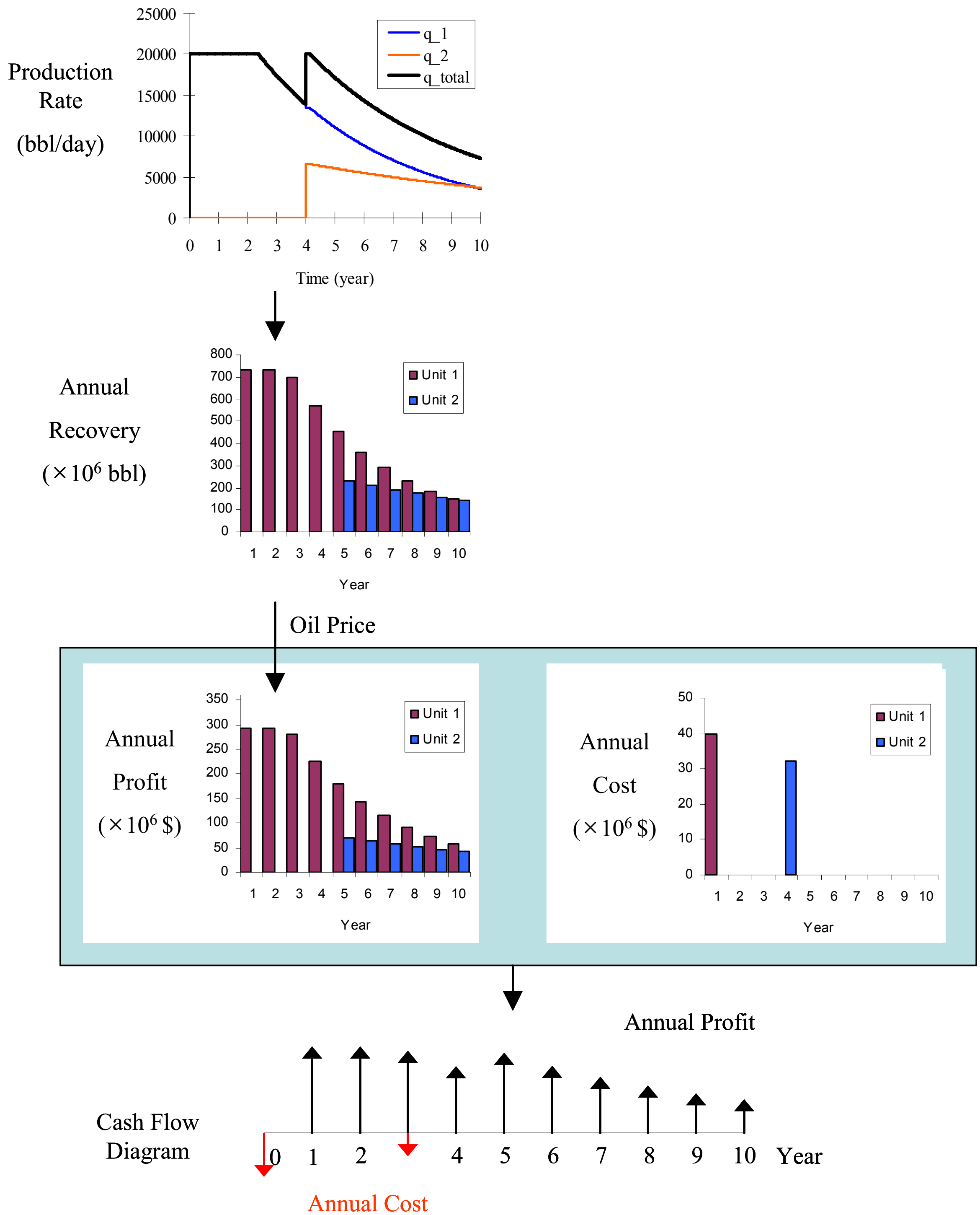


Figure 7.5 Establishing cash flow diagram for economic analysis

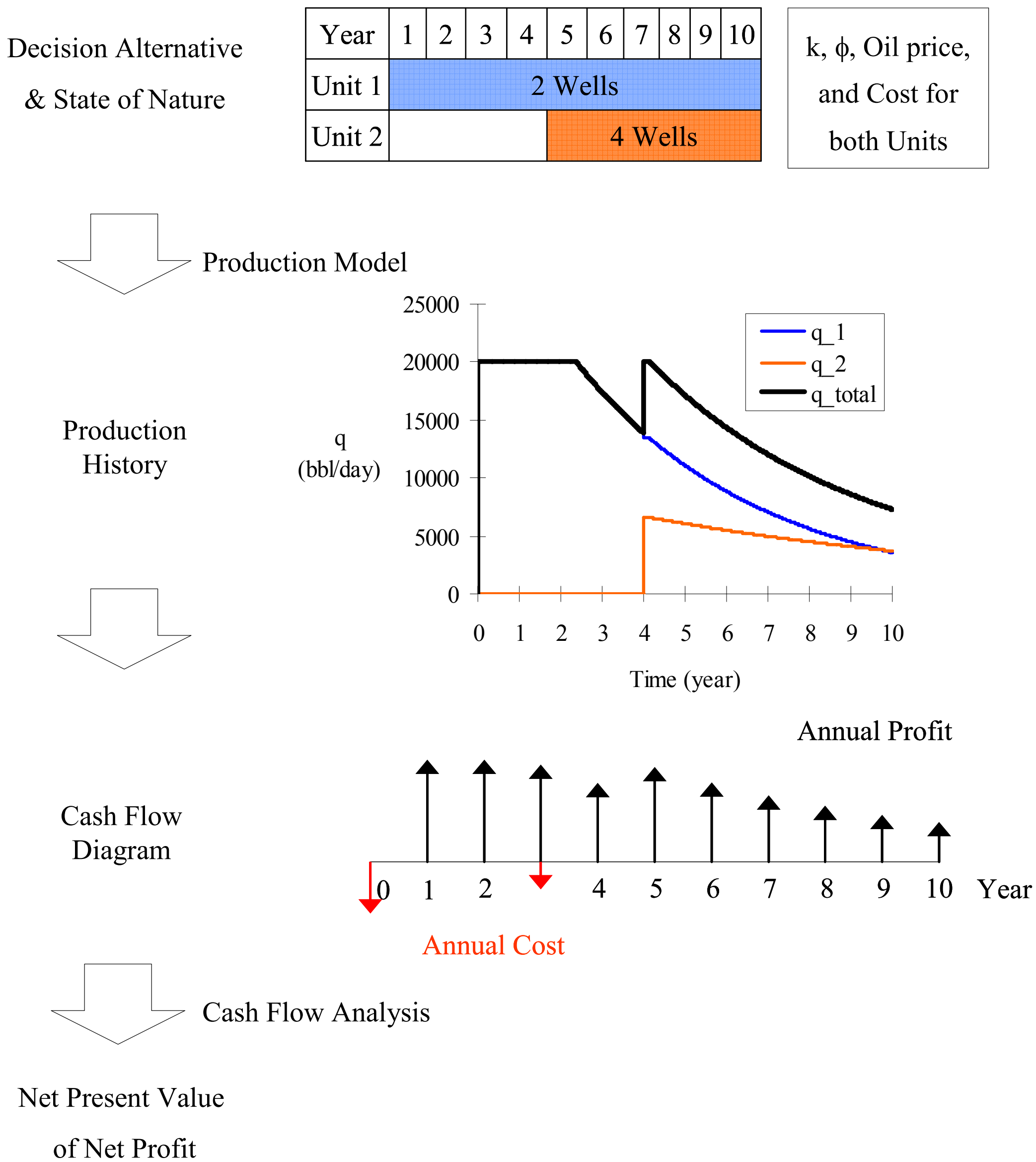


Figure 7.6 Process for calculating NPV of a given decision alternative and a set of uncertain variables

7.2.4 Non-Informative Prior Probabilities

The decision matrix is constructed from the modified tank model and economic analysis based on the decision tree. The number of preference outcomes is 236 in this example. The preference outcomes are shown in Table 7.3. The proposed algorithm provides the non-informative prior probabilities for 50,625 scenarios on the states of nature. Marginal and joint PMFs for the non-informative priors are in Figures 7.7 through 7.10.

While the principle of insufficient reason provides a uniform probability distribution for all cases in Figures 7.7 through 7.10, the decision-based method yields probabilities with an irregular shape. Some have a monotonically increasing or decreasing shape, others are concave, and the others are even more complicated. There is no constraint on the shapes of the probability distributions because the decision-based method is achieving a balance among decision alternatives.

The shape of the PMF is related to the number of preference outcomes corresponding to each bin. If a bin has many corresponding preference outcomes, the non-informative prior probability assigned to the bin is large, because each preference outcome has the same probability. For example, the bin of k_{Unit1}=0.1 (md) includes 10 preference outcomes, and the bin of k_{Unit1}=1,000 (md) has 149 preference outcomes. In a rough estimation, the non-informative prior probability assigned to the latter bin is 15 times as large as the non-informative prior probability for the former. The actual ratio of the non-informative prior probabilities for the latter to the former is 23. The difference between the estimated and actual ratio is caused by the preference outcomes corresponding to both bins k_{Unit1}=0.1 (md) and k_{Unit1}=1,000 (md) and by the difference in the number of states of nature corresponding to each bin. However, the number of preference outcomes provides a good estimation of non-informative prior probabilities, as shown in Table 7.4.

Figures 7.9 and 7.10 show relationships between the physical parameters for both reservoir units. While k and ϕ are generally positively correlated because a larger pore may contribute more space for fluid flow, the correlations in Figure 7.9 are not based on any theories for porous medium - they are intended to contain no information. In other words, the correlation between uncertain variables in non-informative probabilities in this study only captures the relationship to make a decision unbiased. Any available information such as relationships based on data, models and theories is subsequently incorporated through Bayes' theorem.

Table 7.3 Preference outcomes for the production example

<table>
<tr><th>Preference Outcome ID</th><th>Preference Outcome
and the Most Preferred Decision Alternative</th></tr>
<tr><td>1</td><td>Alternative 84 > All other alternatives

Alternative 84
<table>
<tr><td>Year</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Unit 1</td><td colspan="2">1 well</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Unit 2</td><td colspan="8">1 well</td><td></td><td></td></tr>
</table></td></tr>
<tr><td>2</td><td>Alternative 85 > All other alternatives

Alternative 85
<table>
<tr><td>Year</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Unit 1</td><td colspan="2">1 well</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Unit 2</td><td colspan="9">1 well</td><td></td></tr>
</table></td></tr>
<tr><td>⋮</td><td>⋮</td></tr>
<tr><td>236</td><td>Alternative 12200 > All other alternatives

Alternative 12200
<table>
<tr><td>Year</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Unit 1</td><td colspan="10">4 wells</td></tr>
<tr><td>Unit 2</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table></td></tr>
</table>

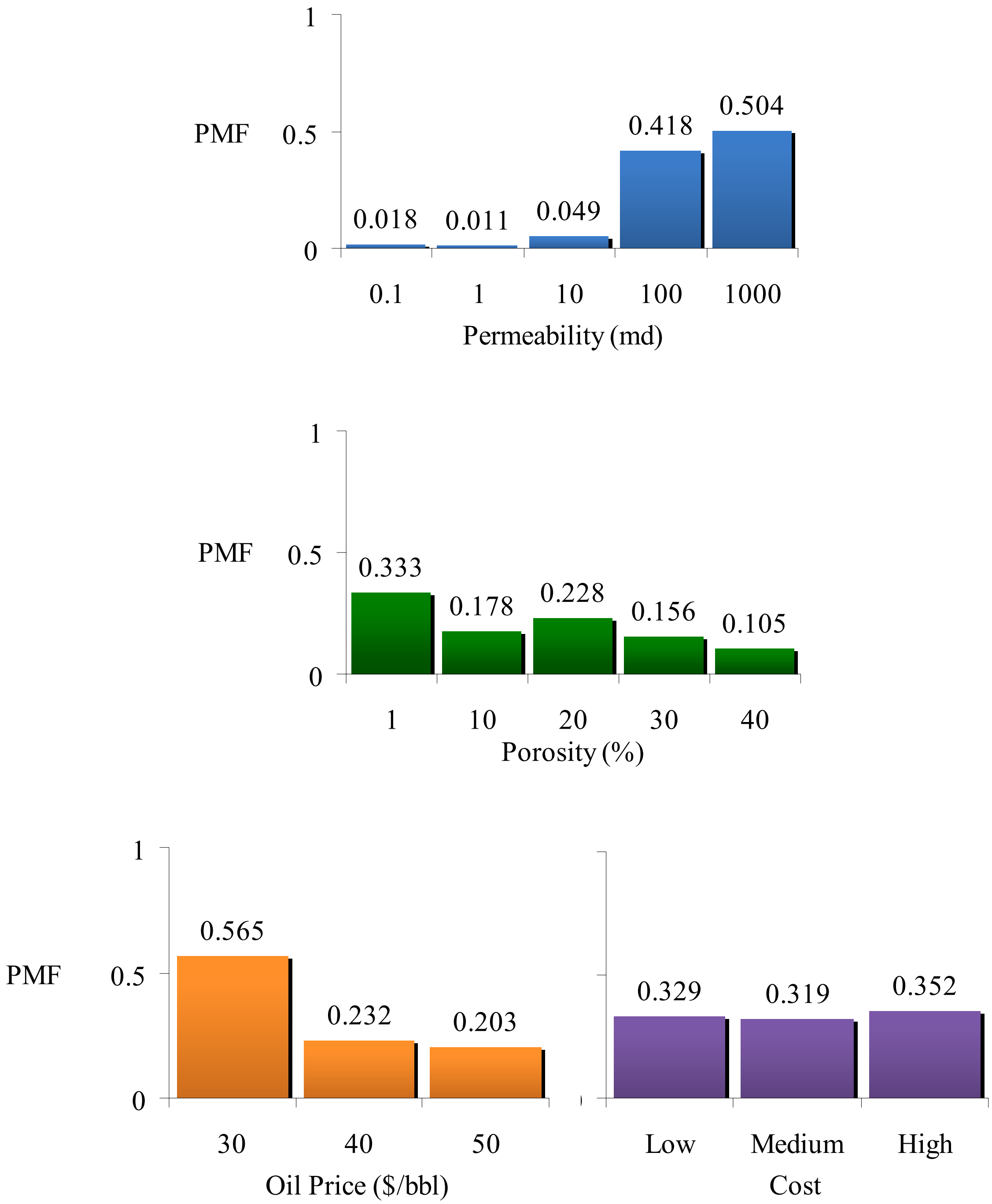


Figure 7.7 Marginal probability distributions for variables of Unit 1

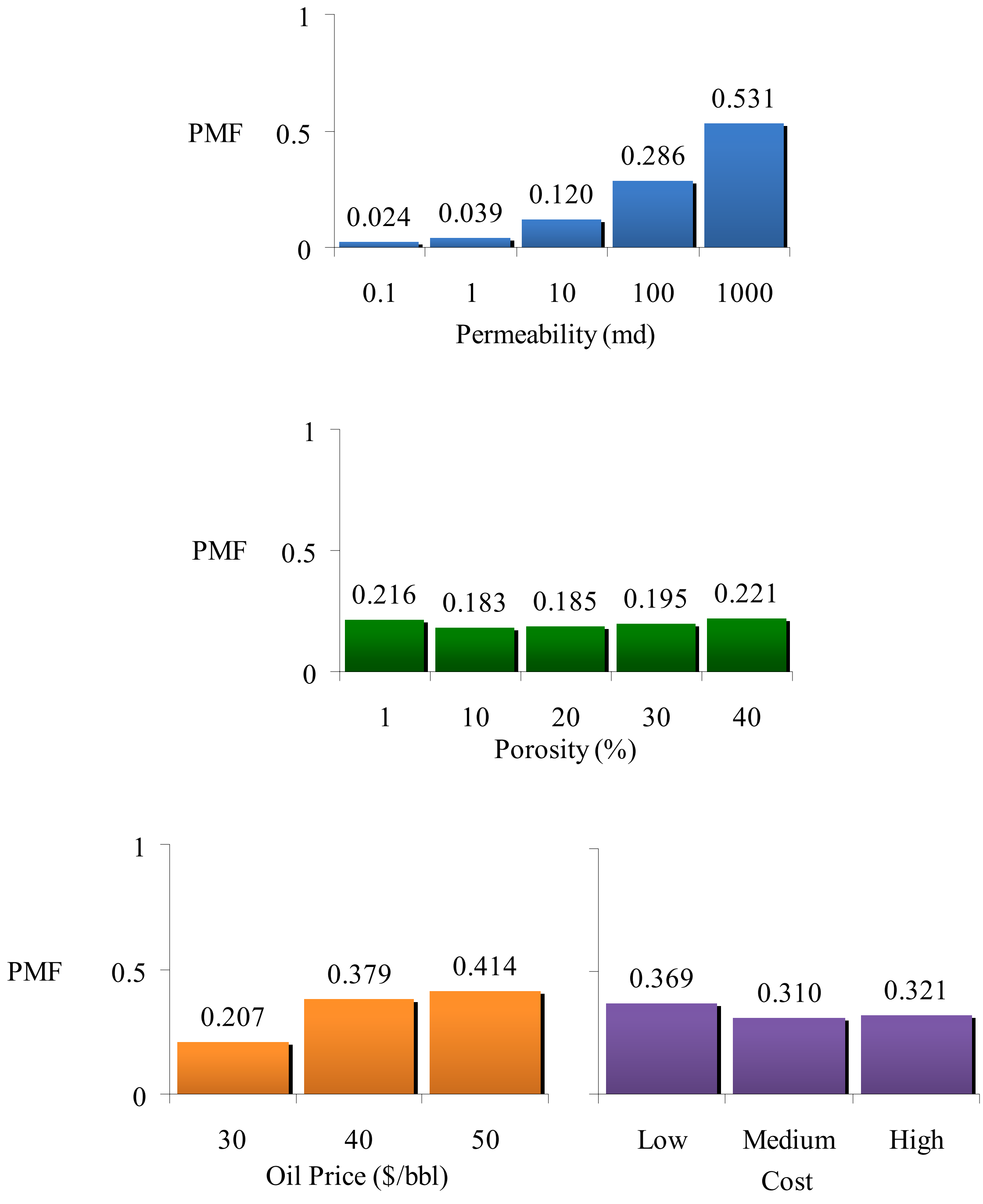
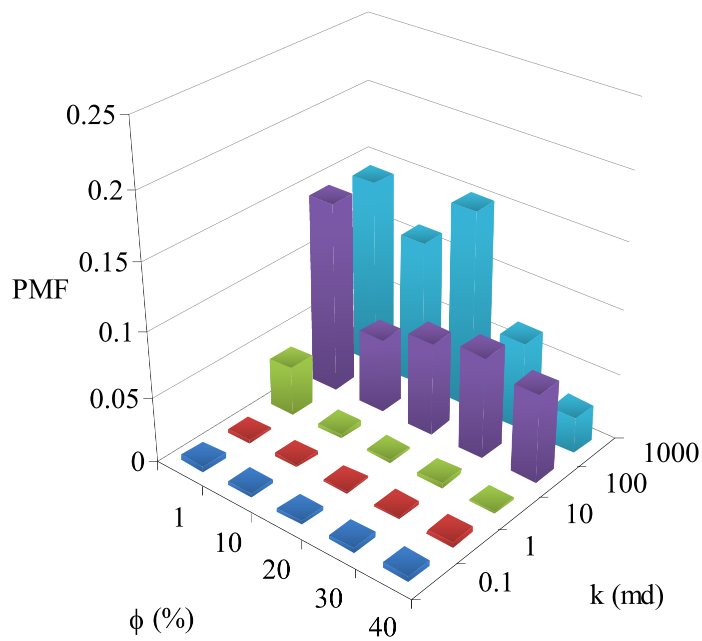
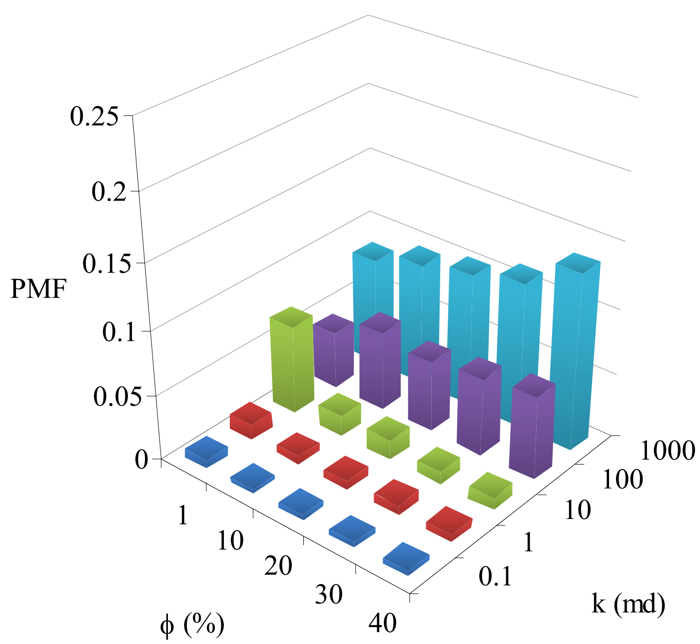


Figure 7.8 Marginal probability distributions for variables of Unit 2



(a) Unit 1



(b) Unit 2

Figure 7.9 Joint probability distributions of porosity and permeability for (a) Unit 1 and (b) Unit 2

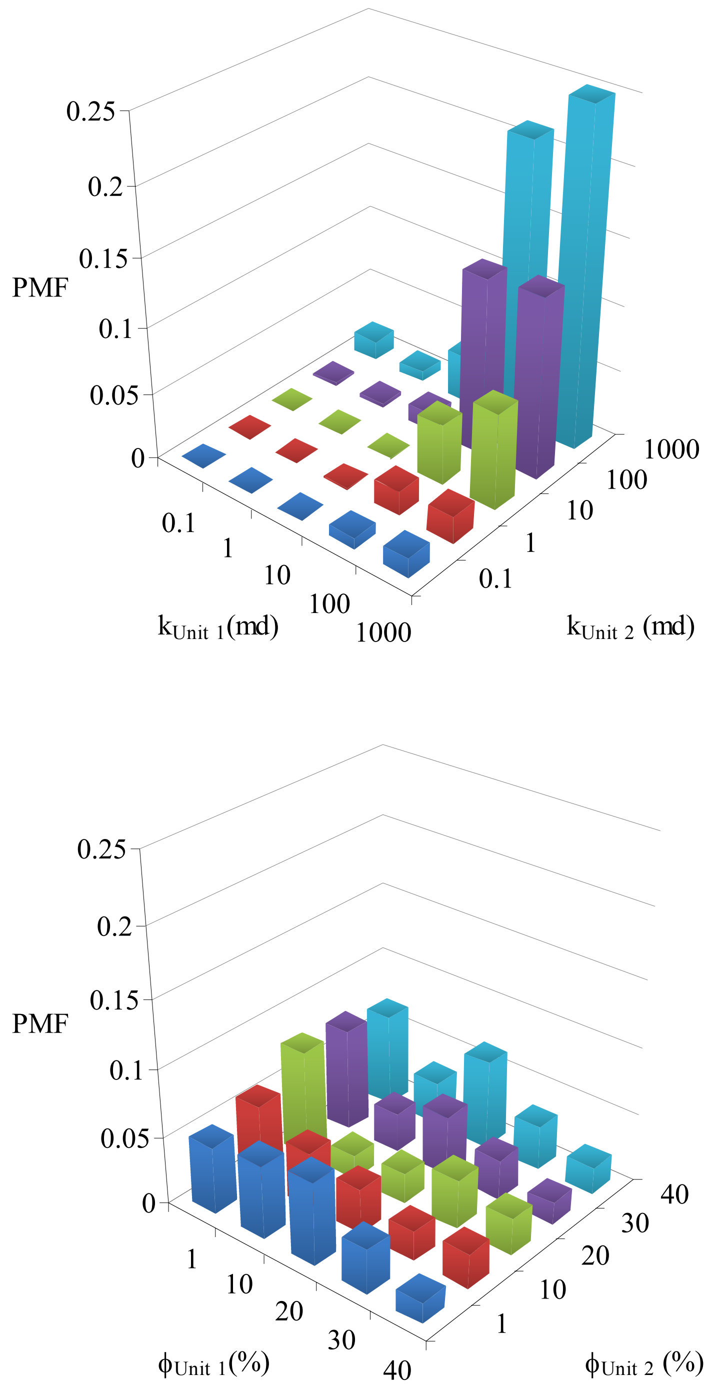
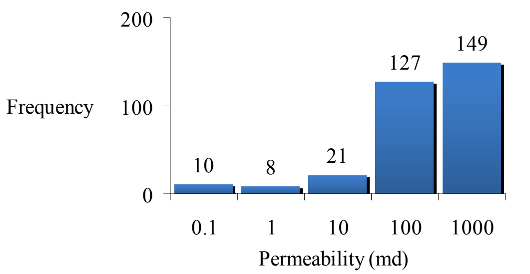
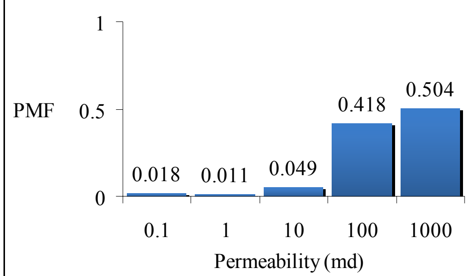
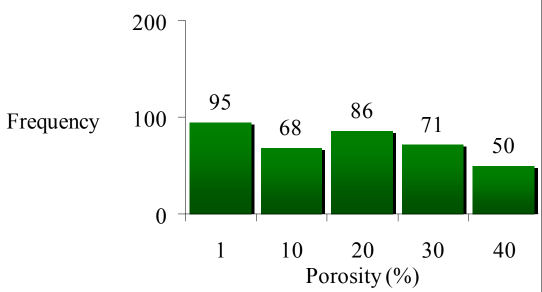
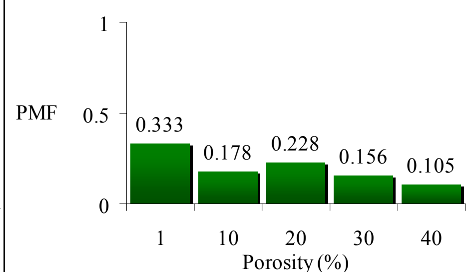
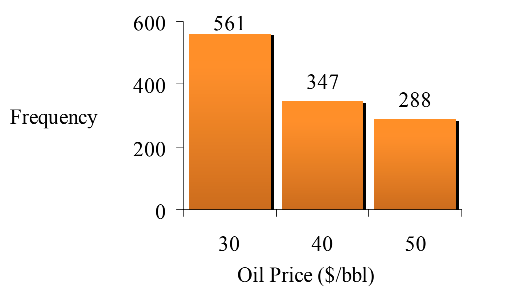
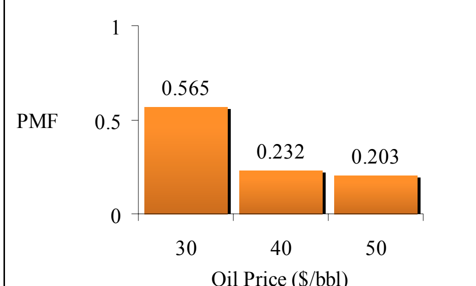
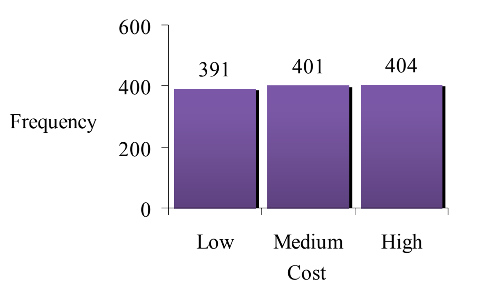
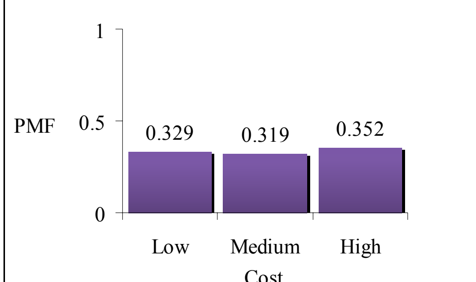


Figure 7.10 Joint probability distributions for permeabilities and porosities of both units

Table 7.4 Relationship between the number of preference outcomes and non-informative prior probabilities

<table>
<tr><th>Variable for Unit 1</th><th>Histogram of the Number of Preference Outcomes</th><th>Marginal PMF</th></tr>
<tr><td>Permeability (md)</td><td>
</td><td>
</td></tr>
<tr><td>Porosity (%)</td><td>
</td><td>
</td></tr>
<tr><td>Oil price ($/bbl)</td><td>
</td><td>
</td></tr>
<tr><td>Cost</td><td>
</td><td>
</td></tr>
</table>

7.2.5 Summary

The first objective of the production example is to show the difference between non-informative prior probabilities obtained from the principle of insufficient reason and the decision-based methods. The non-informative priors are shown in Figures 7.7 through 7.10. The decision-based non-informative prior probabilities emphasize decision outcomes by assigning larger probabilities to states of nature that have unique preference outcomes. In other words, the states of nature that have the same preference outcome have equally distributed probabilities. This assignment is illustrated with Table 7.4.

The second objective is to show that the decision-based method is applicable to practical decision making. The algorithm for decision-based priors was applicable to the production example with 12,241 decision alternatives and 50,625 states of nature.

7.3 TRANSMISSIBILITY EXAMPLE

This example focuses on how the connectivity or transmissibility between reservoir units affects decision making. For example, well placement depends on the degree of connectivity. If the transmissibility is infinite (perfect connectivity), the wells installed in one unit may recover oil from both units. If the reservoir units are separated (zero connectivity), placing wells in both units may be the optimal decision alternative.

7.3.1 Objectives

The transmissibility example accomplishes four objectives. The first objective is to show the importance of heterogeneity associated with decision making in petroleum exploration and production. A parametric study and history matching with an actual oil field will be given to illustrate the significance of characterizing heterogeneity in decision making.

The second objective is to compare the principle of insufficient reason with the decision-based method to obtain non-informative probabilities. The probabilities will illustrate what the probability distribution looks like and why a decision-based non-informative prior is different from that obtained by the principle of insufficient reason.

The third objective is to show how to associate new information on which state of nature is more likely to occur with non-informative prior probabilities. This example will demonstrate how the non-informative prior probabilities are used in practical decision making problems that are usually made with information. The results of a sensitivity analysis will be given to show the influence of information on posterior probabilities, expected utilities for decision alternatives, optimal decision, and the value of perfect information.

The last objective is to show that a different decision making problem may have different non-informative prior probabilities. The example will be given to show the influence of a deterministic parameter, well cost for Unit 1, on non-informative probabilities and on the value of perfect information (VPI).

7.3.2 Reservoir Simulator: Modified Tank Model

A tank-type model provides a simplified but analytical solution. Tank-type models treat the reservoir as a homogeneous porous media. Tank models are useful because they are simple, reducing calculation load, yet realistic enough to simulate reservoir performance based on Darcy's law and mass balance. Walsh and Lake (2003) discussed five variations in tank models. The variations are about the number of layers (single- or multi-layer), interlayer communication (cross flow or no cross flow), and the compressibility of the fluid. Guillot (1999) developed an analytical solution of the tank-type model for the system of two layers in communication. In his model, it is possible to consider both injection and production wells.

In this dissertation, a modified version of the tank model is developed. A schematic diagram of the model is shown in Figure 7.11. The first objective of the new tank model is to introduce a system of two reservoir units with crossflow. The system is basically the same with the tank model for two stratified layers with crossflow. The second objective is to enable a rate-constrained flow. The modified model works for both rate-constrained and pressure-constrained flow periods for high pressure or slightly compressible oil. The modified model simulates depletion flow or pressure during primary recovery.

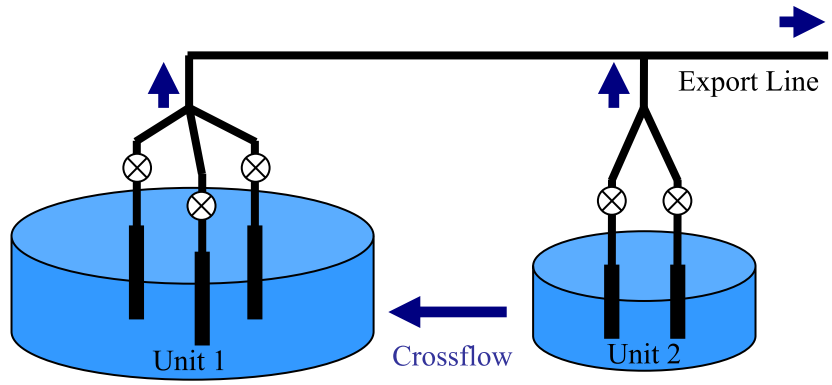


Figure 7.11 Schematic diagram of the modified tank model

The modified tank model is based on the assumptions made in Walsh and Lake (2003) as follows:

1. Three fluid components, stock-tank oil, surface gas, and stock-tank water, are possible.
2. Two fluid phases, oleic and aqueous, are possible.
3. Stock-tank water does not partition into oleic phase.
4. Stock-tank oil does not partition into aqueous phase.
5. Water is immobile.
6. Darcy's law is valid
7. There is no change in temperature.

The modification of tank model is based on the additional assumption:

8. Wellbore pressure is the same for all wells in a unit.
9. The difference between the wellbore pressures in two units is constant.

Assumptions 8 and 9 are from the concept of constant pressure gradient in a wellbore. The reservoir unit is assumed to be thin enough to use a constant wellbore pressure and fluid flow occurs at the same depth for all wellbores. Assumption 9 is based on pressure equilibrium in wellbore and pipeline. Because wellbores in two units are connected through an export line, pressures in production wells interact with each other and ultimately reach a state of equilibrium. In the equilibrium limit, assumption 8 and a constant wellbore pressure gradient should be valid. These constraints lead to a constant pressure difference between wells for each unit.

Mathematically, the modified tank model is based on a solution of a system of nonhomogeneous first-order ordinary differential equations (ODE) with an initial condition. The governing equations include mass balance (Equations 7.1, and 7.2), Darcy's law (Equations 7.3, 7.4, and 7.5) and the equation for a constant pressure difference (Equation 7.6).

$$V_{P1} c_{t1} \frac{d\overline{P}_1}{dt} = -\left(q_1 - q_{XF}\right) \tag{7.1}$$

$$V_{P2} c_{t2} \frac{d\overline{P}_2}{dt} = -\left(q_2 - q_{XF}\right) \tag{7.2}$$

$$q_1 = J_1\left(\overline{P}_1 - P_{wf1}\right) \tag{7.3}$$

$$q_2 = J_2\left(\overline{P}_2 - P_{wf2}\right) \tag{7.4}$$

$$q_{XF} = T\left(\overline{P}_2 - \overline{P}_1\right) \tag{7.5}$$

$$P_{wf1} = P_{wf2} + \Delta P_{wf} \tag{7.6}$$

where V_{p1} and V_{p2} are the pore volume, c_{t1} and c_{t2} are the total compressibility of Unit 1 and 2, q_1 and q_2 are the production rate, J_1 and J_2 are productivity indices, $\overline{P}_1$ and $\overline{P}_2$ are average reservoir pressure, and P_{wf1} and P_{wf2} are wellbore pressure for each unit. q_{XF} is the rate of crossflow and T represents the transmissibility. There are two initial conditions for initial reservoir pressure in each unit.

The solution of the system of ODEs varies with a total production rate, q_T, which is a sum of q_1 and q_2. If q_T is less than or equal to the limit of total production rate, q_{Lim}, P_{wf1} and P_{wf2} are constant. In this case, depletion flow dominates. If q_T is greater than q_{Lim}, q_T is forced to be equal to q_{Lim} and P_{wf1} and P_{wf2} are functions of time. In this situation, a plateau production period occurs. Equations 7.7 through 7.11 show the solution for pressure-specified and Equations 7.12 through 7.16 represent rate-specified flow. Detailed steps in the derivation and solutions for each case are given in Appendix C. The solutions are

$$\overline{P}_1(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \frac{A}{\lambda_1} - \frac{B}{\lambda_2} \tag{7.7}$$

$$\begin{aligned}\overline{P}_2(t) = &\left(\frac{J_1 + T + V_{p1} c_{t1} \lambda_1}{T}\right)\left(C_1 e^{\lambda_1 t} - \frac{A}{\lambda_1}\right) \\ &+ \left(\frac{J_1 + T + V_{p1} c_{t1} \lambda_2}{T}\right)\left(C_2 e^{\lambda_2 t} - \frac{B}{\lambda_2}\right)\end{aligned} \tag{7.8}$$

$$C_1 = \frac{\left(\frac{J_1 + T + V_{p1} c_{t1} \lambda_2}{T}\right)\overline{P}_1(0) - \overline{P}_2(0)}{\left(\frac{V_{p1} c_{t1} \lambda_2 - V_{p1} c_{t1} \lambda_1}{T}\right)} + \frac{A}{\lambda_1} \tag{7.9}$$

$$C_2 = \frac{\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\overline{P}_1(0) - \overline{P}_2(0)}{\left(\frac{V_{p1}c_{t1}\lambda_1 - V_{p1}c_{t1}\lambda_2}{T}\right)} + \frac{B}{\lambda_2} \tag{7.10}$$

$$\begin{aligned}\lambda_{1,2} = \frac{1}{2}\Bigg\{ &-\left(\frac{J_1 + T}{V_{p1}c_{t1}} + \frac{J_2 + T}{V_{p2}c_{t2}}\right) \\ &\pm \sqrt{\left(\frac{J_1 + T}{V_{p1}c_{t1}} - \frac{J_2 + T}{V_{p2}c_{t2}}\right)^2 + 4\left(\frac{T^2}{V_{p1}c_{t1}V_{p2}c_{t2}}\right)}\Bigg\}\end{aligned} \tag{7.11}$$

(Double signs in same order)

$$\begin{aligned}\overline{P}_1(t) = &-\frac{\left(\frac{q_{Lim}}{J_1 + J_2}\right)\left(\frac{J_1 + J_2}{V_{p2}c_{t2}}\right)}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}} + 1\right)} t + C_1 \\ &+ C_2 \exp\left[-\left(\frac{J_1 J_2}{J_1 + J_2} + T\right)\left(\frac{1}{V_{p1}c_{t1}} + \frac{1}{V_{p2}c_{t2}}\right)t\right] \\ &+ \frac{\frac{1}{V_{p1}c_{t1}}\frac{\left(J_1 J_2 \Delta P_{wf} - J_1 q_{Lim}\right)}{J_1 + J_2} + \frac{1}{V_{p2}c_{t2}}\frac{\left(J_1 J_2 \Delta P_{wf} - J_2 q_{Lim}\right)}{J_1 + J_2}}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}} + 1\right)\left(\frac{J_1 J_2}{J_1 + J_2} + T\right)\left(\frac{1}{V_{p1}c_{t1}} + \frac{1}{V_{p2}c_{t2}}\right)}\end{aligned} \tag{7.12}$$

$$\begin{aligned}\overline{P}_2(t) = &-\frac{\left(\frac{q_{Lim}}{J_1 + J_2}\right)\left(\frac{J_1 + J_2}{V_{p2}c_{t2}}\right)}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}} + 1\right)} t + C_1 \\ &- \frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}\left\{C_2 \exp\left[-\left(\frac{J_1 J_2}{J_1 + J_2} + T\right)\left(\frac{1}{V_{p1}c_{t1}} + \frac{1}{V_{p2}c_{t2}}\right)t\right]\right.\end{aligned} \tag{7.13}$$

$$+\frac{\frac{1}{V_{p1}c_{t1}}\frac{\left(J_1J_2\Delta P_{wf}-J_1q_{Lim}\right)}{J_1+J_2}+\frac{1}{V_{p2}c_{t2}}\frac{\left(J_1J_2\Delta P_{wf}-J_2q_{Lim}\right)}{J_1+J_2}}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}+1\right)\left(\frac{J_1J_2}{J_1+J_2}+T\right)\left(\frac{1}{V_{p1}c_{t1}}+\frac{1}{V_{p2}c_{t2}}\right)}\Bigg\}$$

$$C_1=\frac{\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}\overline{P}_1(0)-\overline{P}_2(0)}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}+1\right)} \tag{7.14}$$

$$C_2=\frac{\overline{P}_1(0)-\overline{P}_2(0)}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}+1\right)}$$

$$-\frac{\frac{1}{V_{p1}c_{t1}}\frac{\left(J_1J_2\Delta P_{wf}-J_1q_{Lim}\right)}{J_1+J_2}+\frac{1}{V_{p2}c_{t2}}\frac{\left(J_1J_2\Delta P_{wf}+J_2q_{Lim}\right)}{J_1+J_2}}{\left(\frac{V_{p1}c_{t1}}{V_{p2}c_{t2}}+1\right)\left(\frac{J_1J_2}{J_1+J_2}+T\right)\left(\frac{1}{V_{p1}c_{t1}}+\frac{1}{V_{p2}c_{t2}}\right)} \tag{7.15}$$

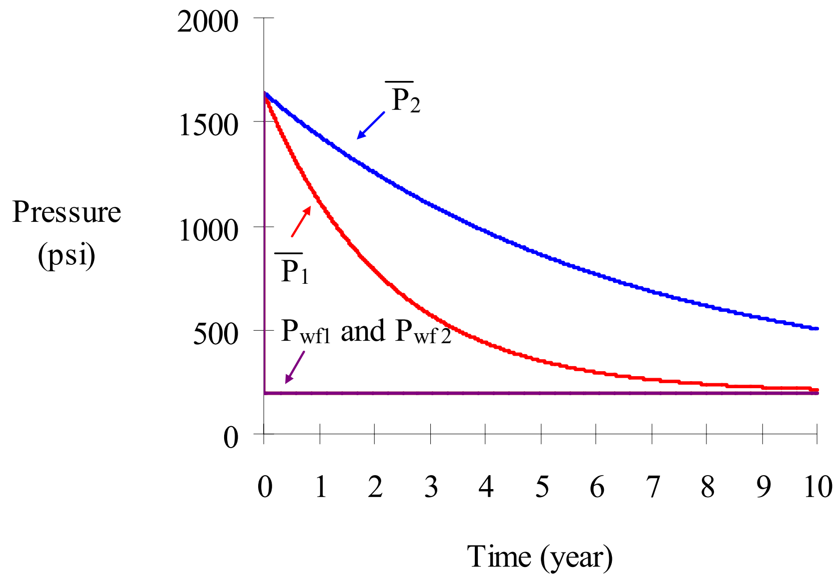
For flexible handling in application, this production model is used with a set of time increments. The time increments enable input of different values for decision parameters or uncertain variables. For example, the number of wells for a certain unit and the productivity index can vary with the increments. If a decision maker wants to increase the number of wells with time, the system of time increments would be 1 well for Year 1, 2 wells for Year 2, 3 wells for Year 3, and so on. The resultant average reservoir pressures of one increment go into the input (initial) condition for the subsequent increment.

Figures 7.12 through 7.14 show example cases of the modified tank model. It is assumed in these examples that the number of wells for each unit does not change during the production life. Input parameters are assumed, as shown in Table 7.5. The first case in Figure 7.12 shows depletion flow without a plateau because the total oil production, q_{total}, which is the sum of the production from both units, is not limited. As shown in Figure 7.12, reservoir pressures for both units and their production rates and the total production rate decay exponentially. The wellbore pressures for wells in both units are constant during the production life. Because there is no communication between units and the wellbore pressures for the coupled wells keeps constant, the behavior of the system of units is equivalent to the sum of the behavior of two separate units.

In the second case, the total production is limited. As shown in Figure 7.13, the limit causes a plateau period in an early stage of production life. During the plateau period, wellbore pressures decrease linearly. While the total production rate is constant during the plateau period, the production rates for individual units are not constant. The reason for this difference in production rates is that there is no connectivity between units. In the case of the separated units, the difference in productivity index causes different rates of decay in average reservoir pressure. Therefore, the difference in reservoir average pressure and wellbore pressure varies with time. The production rates for the two units are not constant during the plateau period. The production rates would be constant if there is a large degree of communication between units, as shown in Figure 7.14. The communication enables equality in average reservoir pressure for both units, and the average reservoir pressure decreases almost linearly with time. In fact, if the communication is perfect, the two units behave like one single reservoir and have identical pressure histories. The difference in production rates for units is caused by the difference in productivity index.

Table 7.5 Input parameters in three example cases in Figures 7.12 through 7.14

Parameter		Example 1 (Figure 7.12)	Example 2 (Figure 7.13)	Example 3 (Figure 7.14)
Drainage area	(acres)	1,000 (Unit 1) 500 (Unit 2)		
Net pay thickness	(ft)	150		
Porosity	(%)	31		
Permeability	(md)	30 (Unit 1) 5 (Unit 2)		
Total compressibility	(1/psi)	0.00008		
Initial reservoir pressure	(psi)	1,640		
Designated wellbore pressure	(psi)	199		
Oil viscosity	(cp)	0.35		
Maximum production rate	(STB/day)	∞	25,000	25,000
Transmissibility	(RB/psi/day)	0	0	100



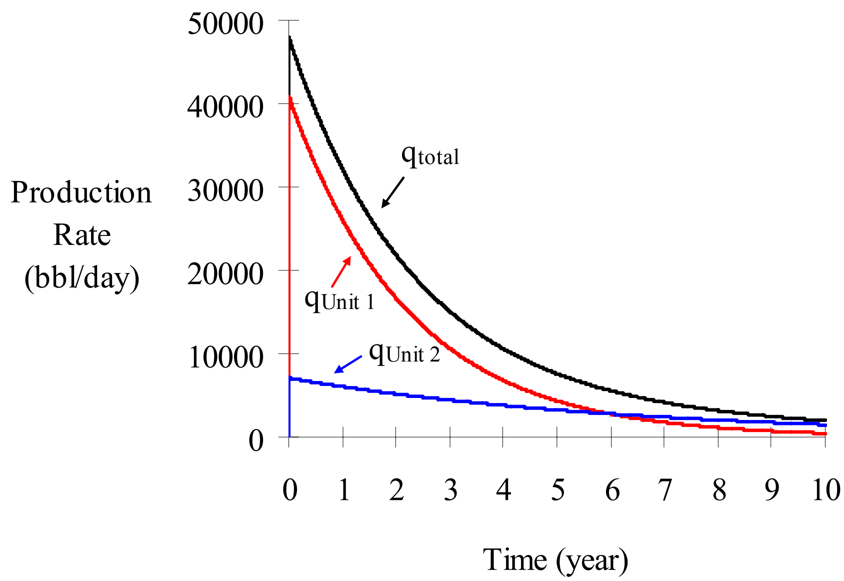
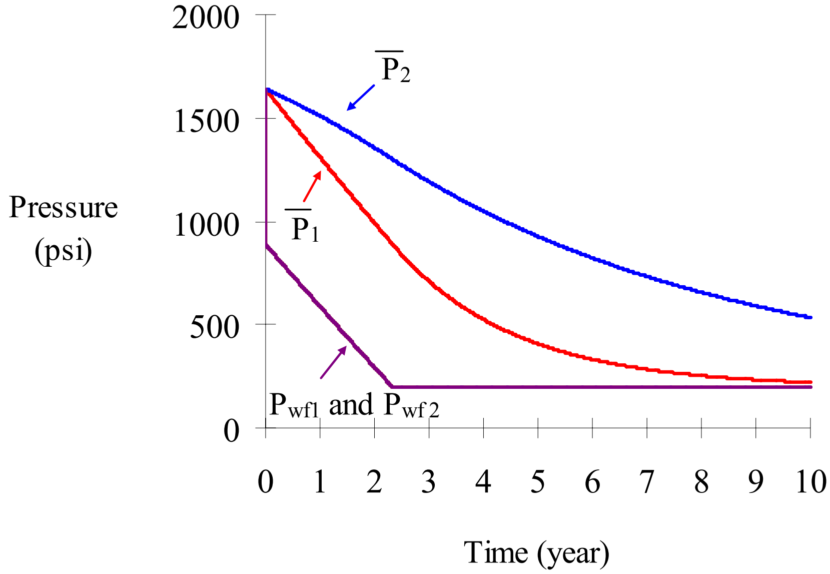


Figure 7.12 Simulation results with no production rate limit and no communication between the two units



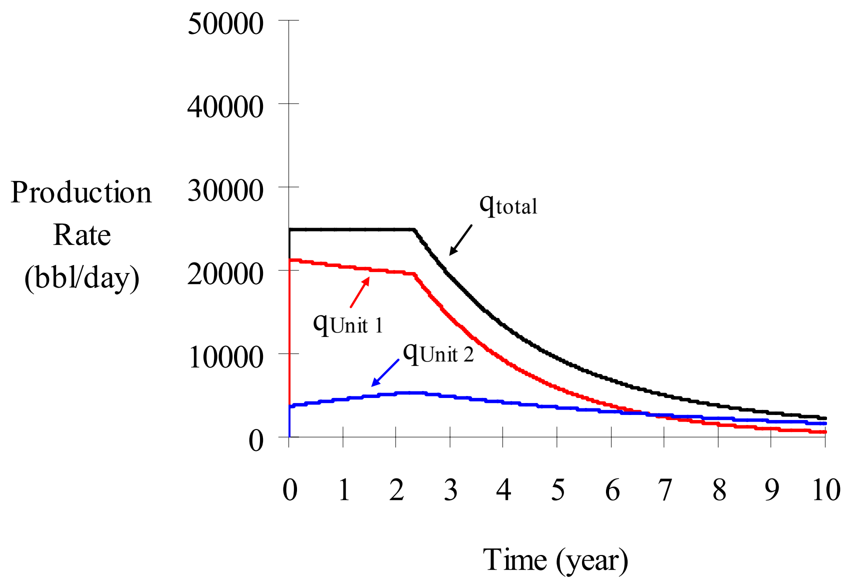


Figure 7.13 Simulation results with production rate limit and without communication between the two units

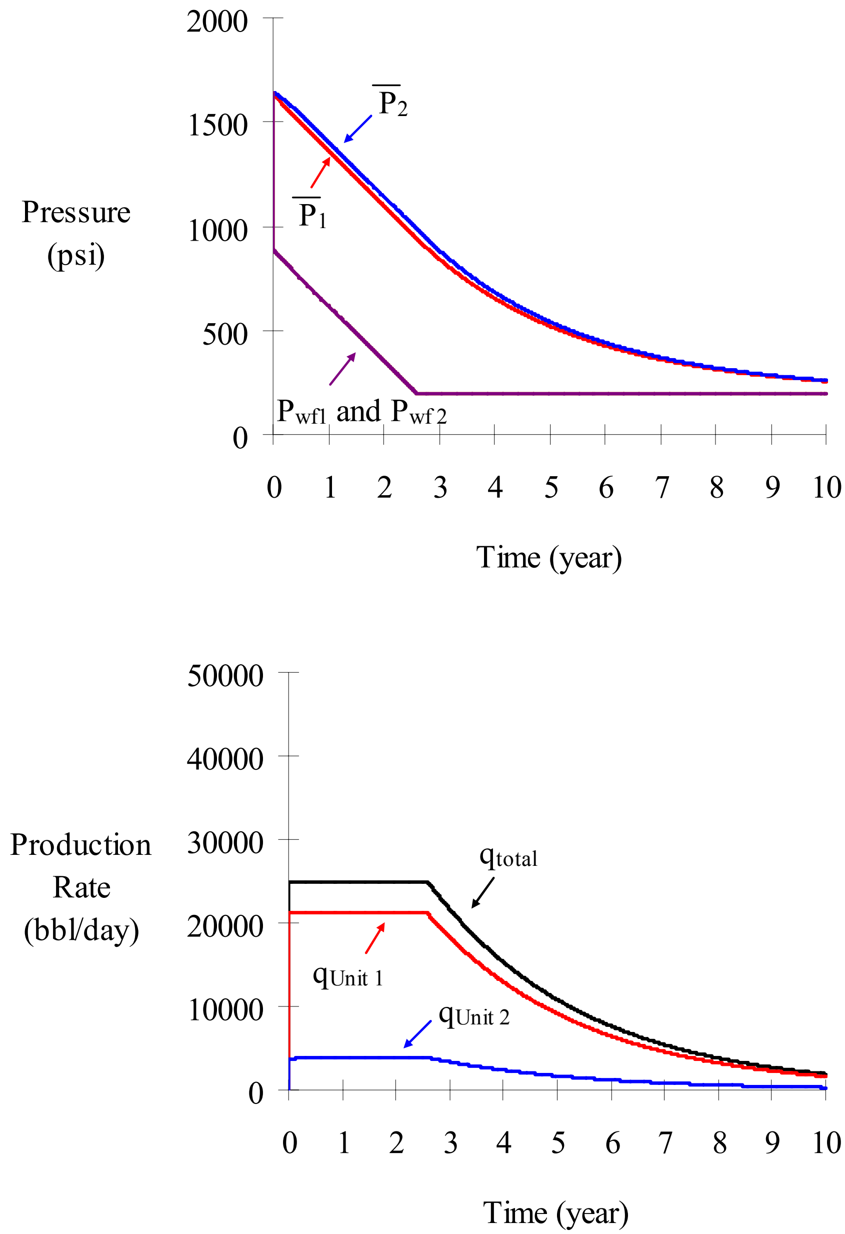


Figure 7.14 Simulation results with production rate limit and large communication between two units

7.3.3 Communication between Production Units

Connectivity between production units is an important consideration in predicting well performance. The connectivity or communication is quantified by the transmissibility. The transmissibility is defined by Equation 7.16 (Guillot, 1999).

$$T = \frac{kA}{\mu w} \tag{7.16}$$

The transmissibility is a function of a porous media property (the permeability, k), the geometry of the porous media (the area of the cross section, A, and the length of flow, w), and a fluid property (the viscosity of the fluid, μ). If connectivity is perfect - in other words, infinite transmissibility - two production units can be considered a single unit. If there is no connectivity, the transmissibility is equal to zero and the two production units are totally separated except at the surface. The model used in the modified tank model assumes that there is a thin transition zone between two production units. The properties in Equation 7.16 - k, A, and w - are considered parameters of the transition zone.

Consider a simple production case to understand the production performance for both extremes of transmissibility. Figure 7.15 illustrates a schematic diagram for oil production from a well in each unit. Two representative factors, reservoir pore volume (V_p) and productivity index (J) for a well, characterize well production in each unit above. The flow, q_{XF}, indicates crossflow caused by a pressure difference between the two units. With a simple tank-type production model, the production rate histories are obtained by the following equations:

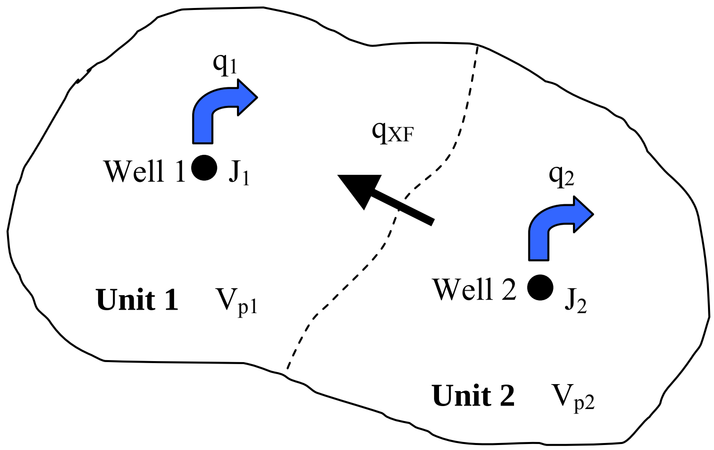


Figure 7.15 Schematic diagram of possibly compartmentalized reservoir

For T=0,

$$\lambda_1 = \frac{J_1}{V_{p1} c_t} \tag{7.17}$$

$$\lambda_2 = \frac{J_2}{V_{p2} c_t} \tag{7.18}$$

$$q_1 = J_1 \Delta p_{ini} \exp(-\lambda_1 t) \tag{7.19}$$

$$q_2 = J_2 \Delta p_{ini} \exp(-\lambda_2 t) \tag{7.20}$$

$$q_{T=0} = q_1 + q_2 \tag{7.21}$$

For T=∞ case, two units are in the same tank-type reservoir. Therefore, the average pressures of both units are equal to each other.

$$\lambda_{T=\infty} = \frac{J_1 + J_2}{(V_{p1} + V_{p2}) c_t} \tag{7.22}$$

$$q_{T=\infty} = (J_1 + J_2)\Delta p_{ini} \exp(-\lambda_{Total} t) \tag{7.23}$$

where total compressibility, c_t, and initial pressure difference between reservoir and wellbore, Δp_{ini}, are kept constant. The parameter, λ, is the decay constant, which indicates the rate of depletion.

The total production from both wells in this example is shown in Figure 7.16. The first case is when both units have the same pore volume and productivity indices. The total well production for T=0 and T=∞ are identical. For the other cases, the difference in productivity index or reservoir pore volume causes different total production, depending on the transmissibility. The production rate when there is no connectivity decreases faster with time than that when there is perfect connectivity. In the second case of J_1=J_2 and V_{p1}>V_{p2}, the total oil recovery of T=0 is less than that of T=∞ at an early stage and becomes greater at a later stage. These changes of production rate at later stages do not necessarily make larger benefit because large production in the early stages is generally most preferred by a decision maker. This factor is considered in the economic (consequence) analysis where the time value of money is included.

The simple theoretical example above illustrates the potential importance of the connectivity between production units. The connectivity influences the well behavior, and accordingly, the total oil recovery and net profit. If the connectivity is ignored and perfect communication is assumed, the net profit might be overestimated and a decision might mislead a development strategy.

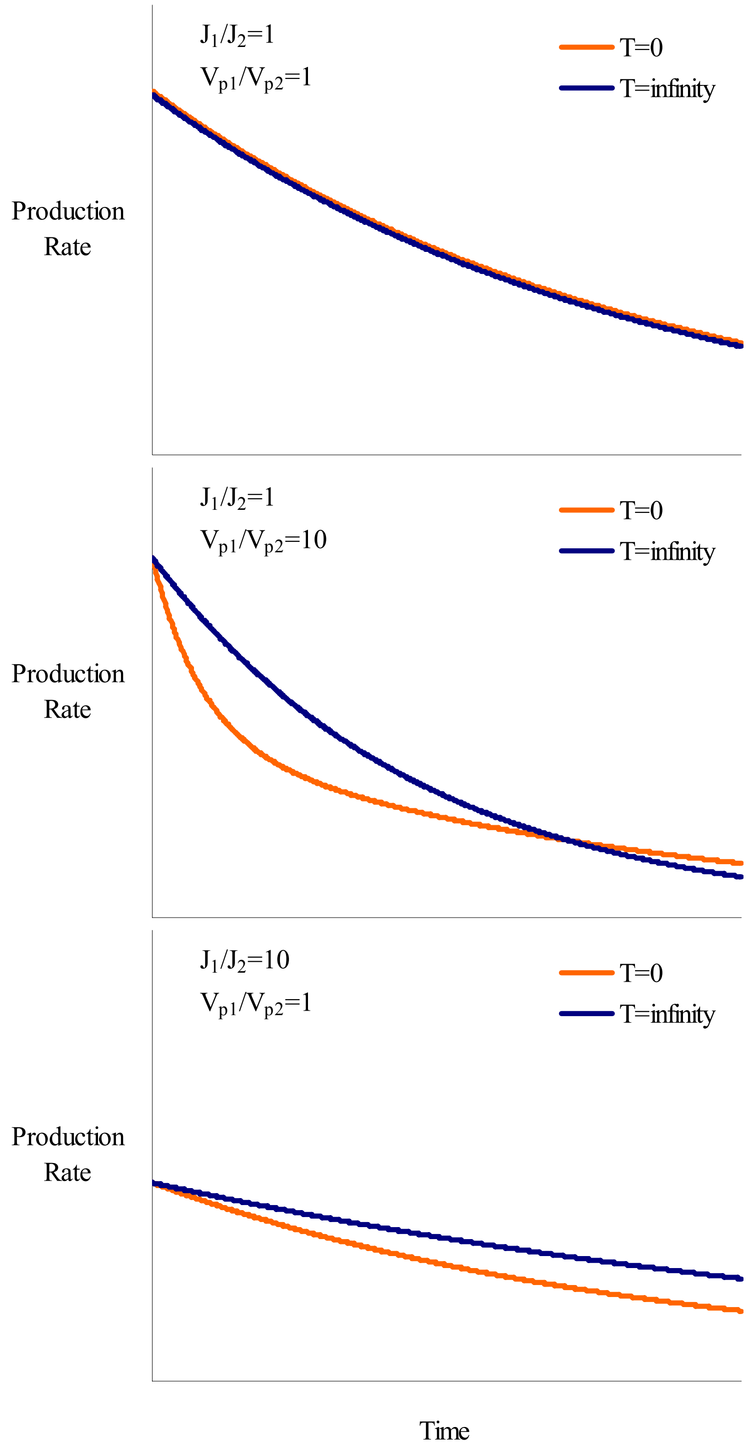


Figure 7.16 Difference in well behavior for extreme cases of well connectivity under three circumstances

The importance of connectivity can also be illustrated by a real case history. Production data of two wells in BP's Holstein oil field was obtained from the Minerals Management Service (MMS) (2008). Monthly oil production and the production per month were computed and analyzed for the two wells, A005 and A006, that were thought to be located in the same channel (Wiseman *et al*., 2007). The modified tank-type model was used for history matching based on least square error on the monthly oil production. Microsoft Excel Solver is used to find the optimized solution. The detailed procedure and results of the history matching are presented in Appendix A.

History matching results are shown in Figures 7.17 and 7.18. The matching curves show a good match within the early part of production history; after 2½ years, the difference between the data and the model increases. A comparison of the total oil recovery between the data and the model is presented in Table 7.6. As seen in Figure 7.18, the average reservoir pressures for both units are not equal with time. This means that the connectivity between the two units is not perfect. The matching gives transmissibility between the units equal to 59 (bbl/psi/day). The estimated transmissibility is larger than the productivity index for both wells - 25 and 49 (bbl/psi/day) for Units 1 and 2, respectively - but not large enough to imply perfect communication between units.

Table 7.6 Comparison of oil recovery from actual data and history matching

	Database	History matching
Unit 1	5.474 (mmbbl)	4.703 (mmbbl)
Unit 2	5.137 (mmbbl)	4.187 (mmbbl)
Total	10.611 (mmbbl)	8.890 (mmbbl)

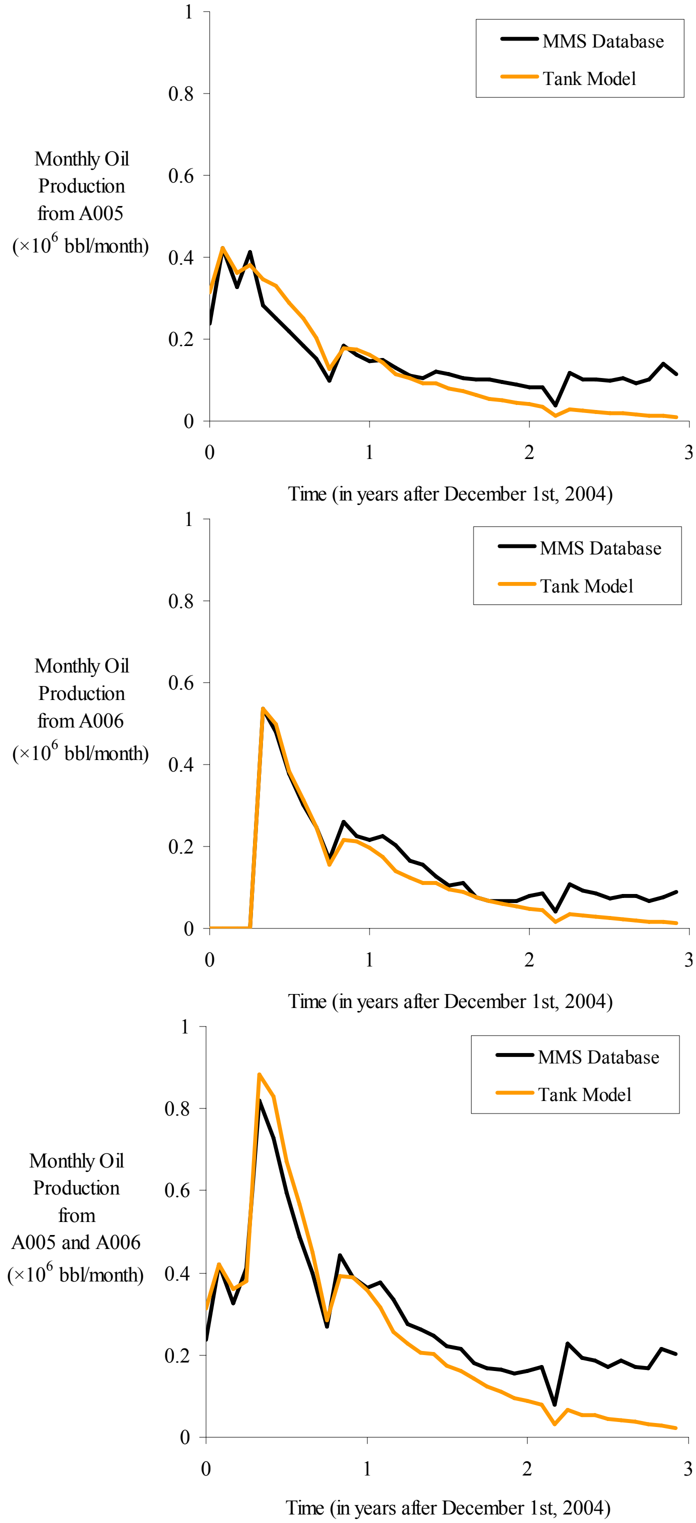
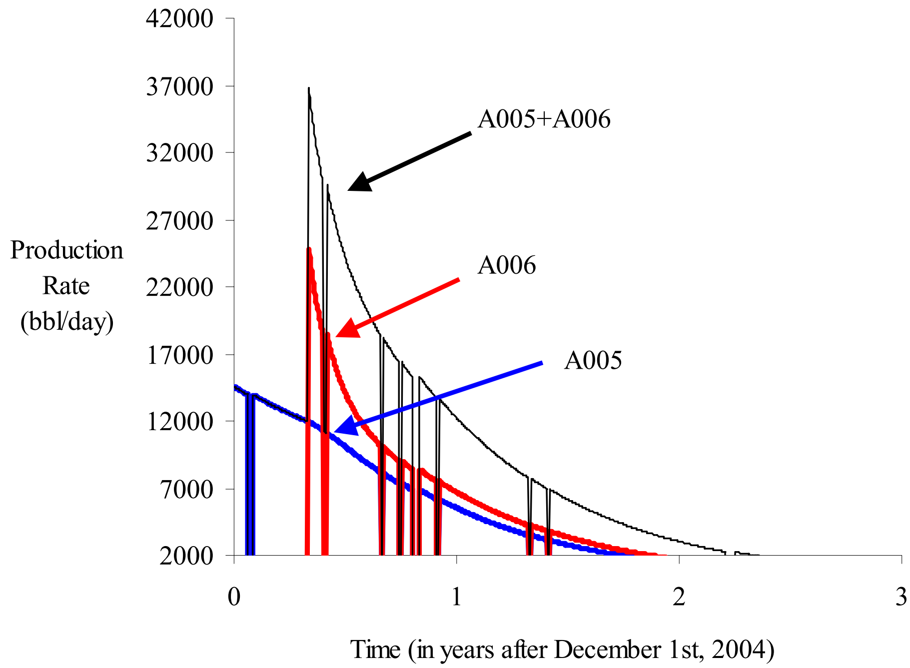


Figure 7.17 History matching between production data from MMS and regression results



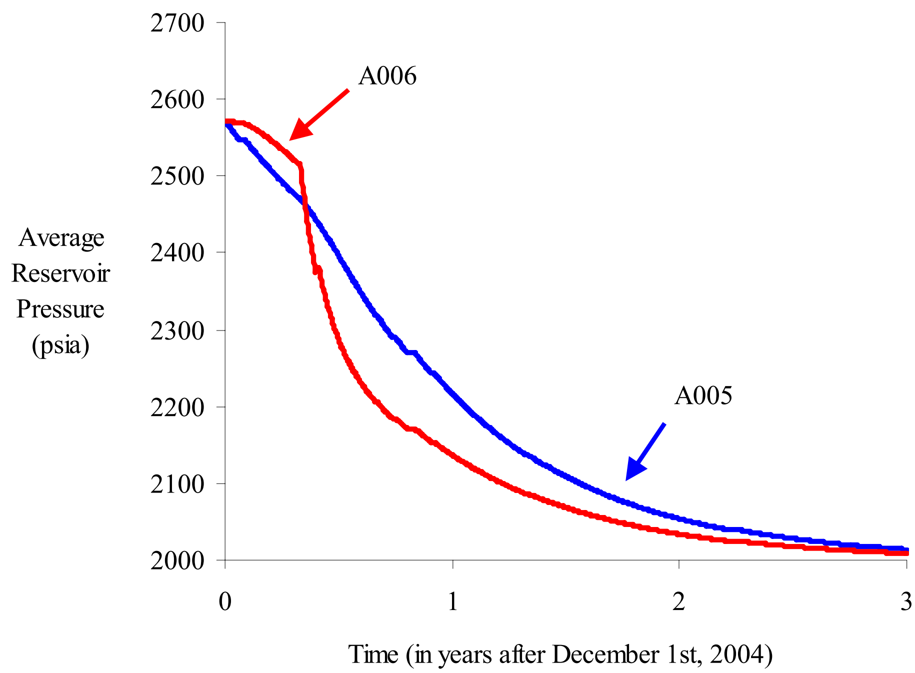
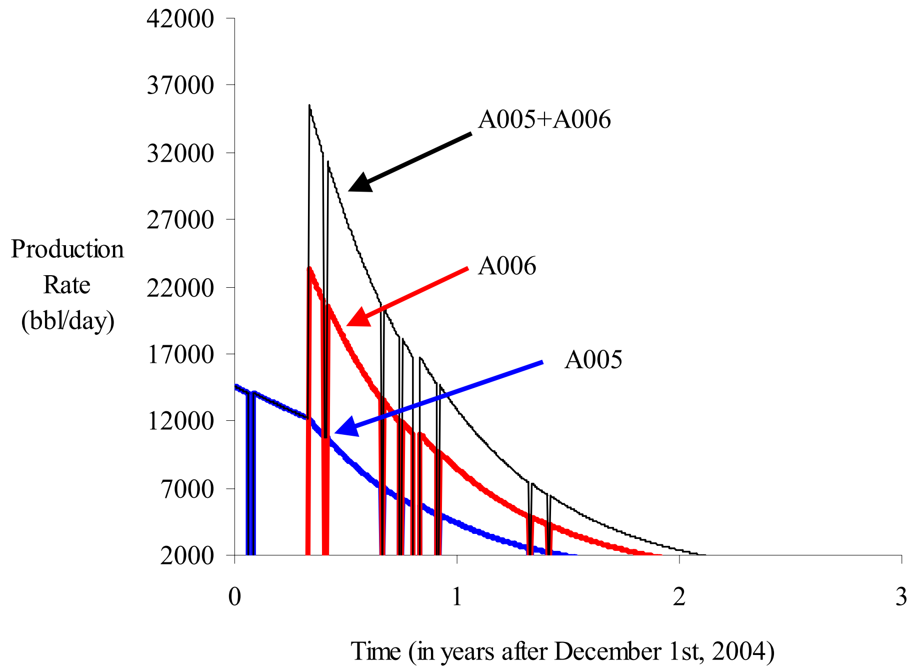


Figure 7.18 Time history of production rate and average reservoir pressure obtained by history matching with a tank-type model

If transmissibility is set equal to infinity, the well behavior under perfect connectivity can be obtained. This behavior is what a decision maker would expect if the wells are assumed to be connected. Figures 7.19 and 7.20 show the well behavior under perfect connectivity. The two curves for the average reservoir pressures versus time are identical in this case. Oil production under perfect connectivity was greater than that found from history matching (Table 7.7). The difference between total oil productions is 105,040 bbl. If oil price is $100/bbl, the monetary difference is equal to $10.5 MM. Therefore, the assumption of the perfect connectivity might affect a decision about how to produce from this reservoir.

Table 7.7 Comparison of oil recovery from history matching case and perfect connectivity case

	History matching (T=59 bbl/psi/day)	History matching + T→∞
Unit 1	4.703 (mmbbl)	4.094 (mmbbl)
Unit 2	4.187 (mmbbl)	4.901 (mmbbl)
Total	8.890 (mmbbl)	8.995 (mmbbl)



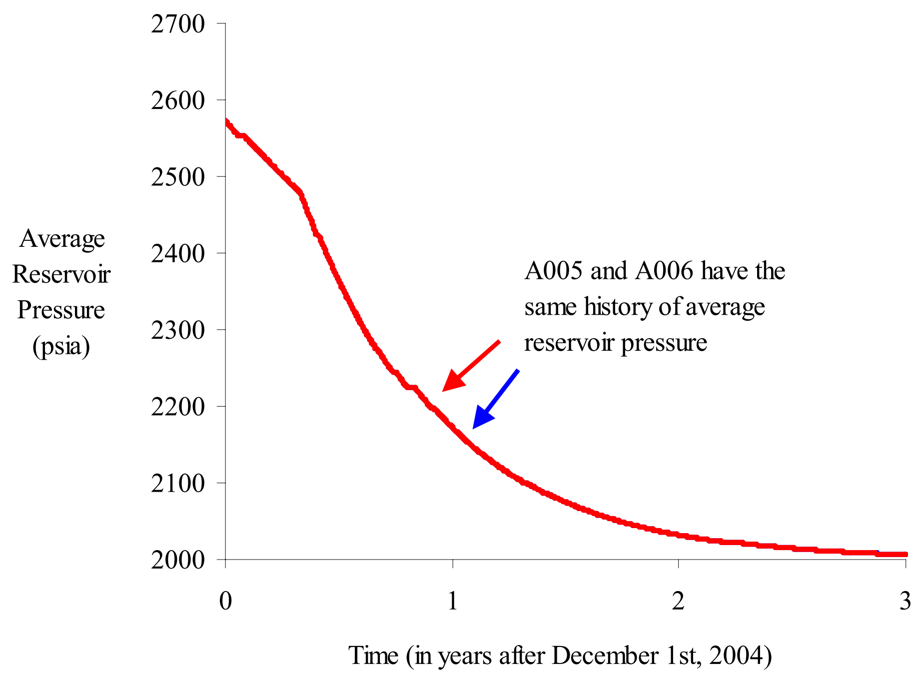


Figure 7.19 Time history of production rate and average reservoir pressure obtained by forcing perfect connectivity with a tank-type model

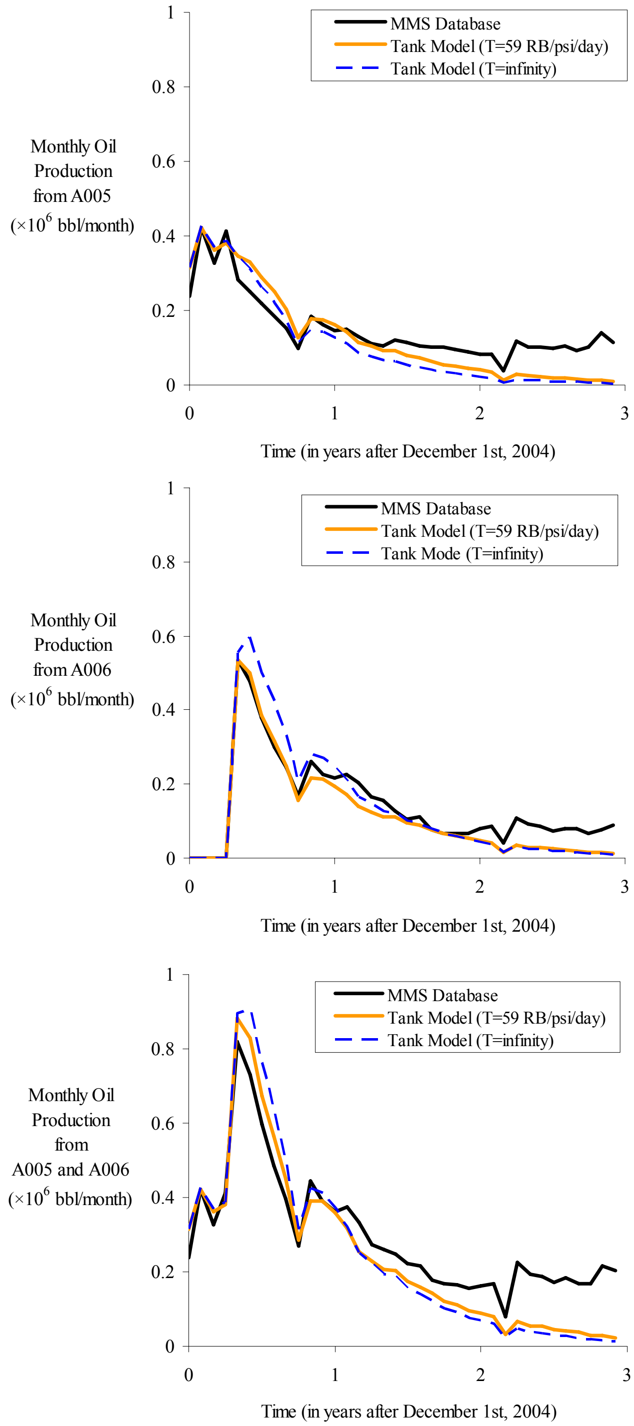


Figure 7.20 Comparison of oil production histories from database (unknown transmissibility), history matching (back-calculated connectivity), and a tank model (perfect connectivity)

7.3.4 Decision Description

A decision maker designs production schedules based on primary recovery from the field. The oil field consists of two reservoir units, as shown in Figure 7.21. It is assumed that Unit 1 has larger productivity index, pore volume, permeability, and cost compared to Unit 2. If the transmissibility is large - that is, two units are not separated and in perfect communication - it is not necessary to install wells in Unit 1, because oil in Unit 1 can be recovered from Unit 2 with less cost. In the other extreme, no communication between units, a decision maker may place wells in both units if the production from individual units makes the net profit greater than zero.

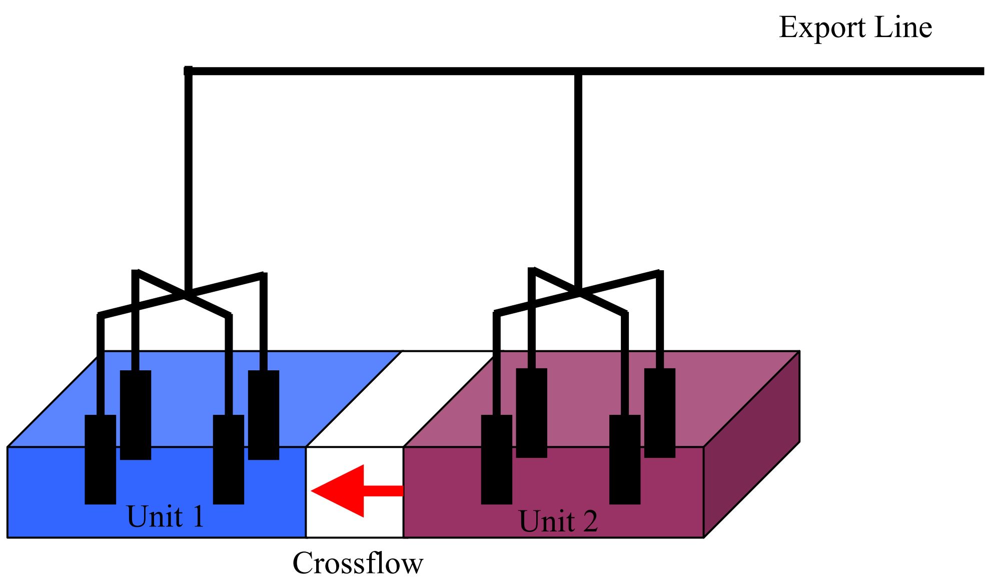


Figure 7.21 Reservoir units with communication

There are many possible decision alternatives. In this example, the maximum number of wells for the individual unit is limited to 4, and the production period in scope is 10 years. At least one of units must be in production during the period. For example, it is not allowed to close all valves of production wells in both units for any time period. The number of wells for each unit is set constant during production. From these assumptions, a decision maker ends up with 12,241 decision alternatives, including a “No go” option, which means that the field is abandoned if unprofitable. The set of decision alternatives is the same one used in the previous production example. The only uncertain variable in this example is the transmissibility between reservoir units. The transmissibility is discretized into 12 possible bins with 0, 0.1, 0.2, 0.4, …, 51, and 102 (RB/psi/day). The other parameters are assumed to be deterministic – their values are shown in Table 7.8.

Table 7.8 Deterministic parameters in transmissibility example

Parameter		Unit 1	Unit 2
Drainage area	(acres)	1,000	500
Net pay thickness	(ft)	50	50
Porosity	(%)	20	20
Permeability	(md)	10	5
Total compressibility	(1/psi)	0.00005	0.00005
Initial reservoir pressure	(psi)	2,150	2,150
Designated wellbore pressure	(psi)	1,500	1,500
Oil viscosity	(cp)	0.8	0.8
Discount rate	(%)	5	5
Oil price	($/bbl)	40	40
Facility cost	($ MM/unit)	10	5
Well cost	($ MM/well)	4	2
Maintenance cost	($ MM/year)	1	0.5
Maximum production rate	(STB/day)	1500	

7.3.5 Decision Framework

The decision with 12,241 alternatives and 12 states of nature is illustrated with a decision tree in Figure 7.22. Table 7.9 shows the most preferred alternative for each state of nature. For small transmissibility, such as states 1 through 5, the most preferred alternatives represent oil production from both of the reservoir units. For states of larger transmissibility, the only unit in production is Unit 1. This means that the production from Unit 1 is profitable because of a larger productivity index than Unit 2. If the transmissibility is much larger, the production from Unit 2 is profitable, because wells in Unit 2 are capable of producing the oil in Unit 1 with less cost.

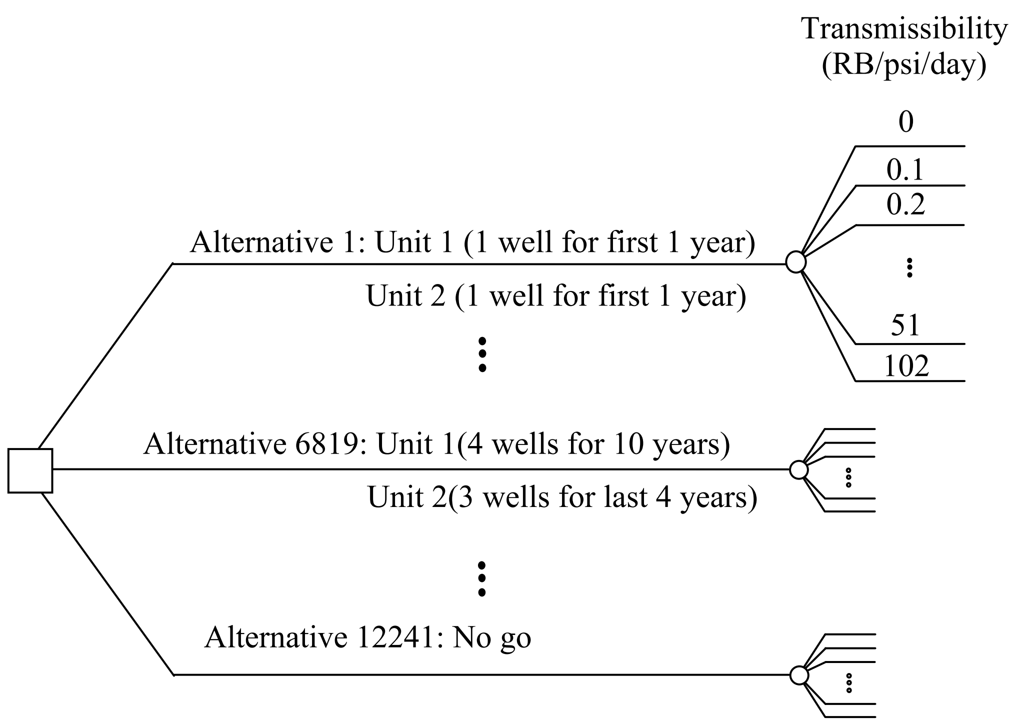


Figure 7.22 Decision tree for transmissibility example

Table 7.9 The most preferred decision alternative for each state of nature in the transmissibility example

<table>
<tr><th colspan="2">State of nature</th><th rowspan="2">The Most Preferred Decision Alternative</th></tr>
<tr><th>Number</th><th>Transmissibility, T (RB/psi/day)</th></tr>
<tr><td>1</td><td>0</td><td>Alternative 3285
<table><tr><td>Year</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr><tr><td>Unit 1</td><td colspan="10">2 wells</td></tr><tr><td>Unit 2</td><td colspan="10">2 wells</td></tr></table></td></tr>
<tr><td>2</td><td>0.1</td><td rowspan="4">Alternative 3340
<table><tr><td>Unit 1</td><td colspan="10">2 wells</td></tr><tr><td>Unit 2</td><td colspan="10">3 wells</td></tr></table></td></tr>
<tr><td>3</td><td>0.2</td></tr>
<tr><td>4</td><td>0.4</td></tr>
<tr><td>5</td><td>0.8</td></tr>
<tr><td>6</td><td>1.6</td><td>Alternative 12190
<table><tr><td>Unit 1</td><td colspan="10">3 wells</td></tr><tr><td>Unit 2</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr></table></td></tr>
<tr><td>7</td><td>3.2</td><td rowspan="6">Alternative 12240
<table><tr><td>Unit 1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr><tr><td>Unit 2</td><td colspan="10">4 wells</td></tr></table></td></tr>
<tr><td>8</td><td>6.4</td></tr>
<tr><td>9</td><td>12.8</td></tr>
<tr><td>10</td><td>25.6</td></tr>
<tr><td>11</td><td>51.2</td></tr>
<tr><td>12</td><td>102.4</td></tr>
</table>

7.3.6 Non-Informative Prior Probabilities

The number of preference outcomes is 4 in this example. The preference outcomes are:

Alternative 3285 > (All other decision alternatives)

Alternative 3340 > (All other decision alternatives)

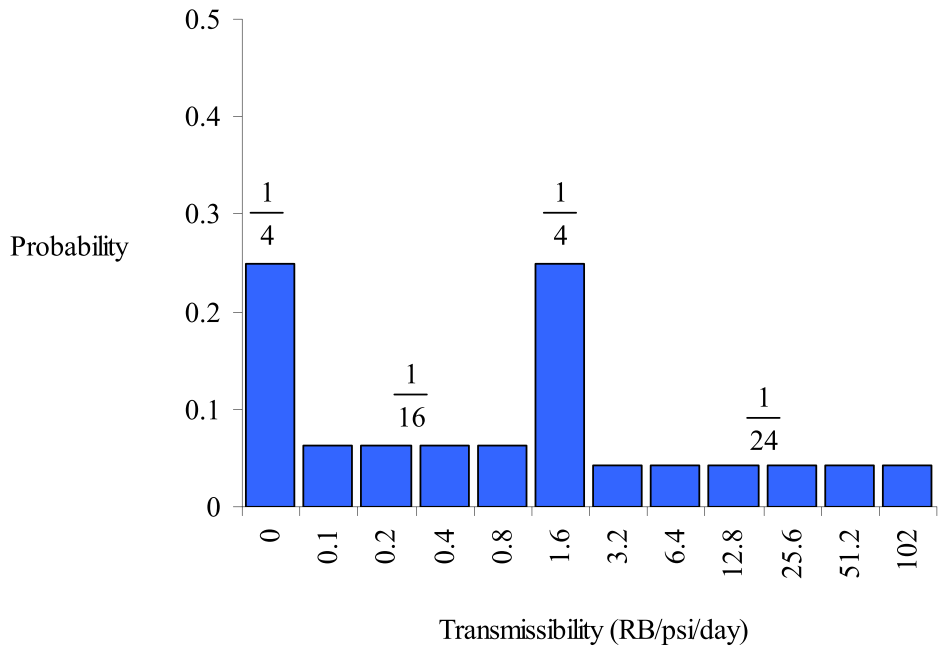
Alternative 12190 > (All other decision alternatives)

Alternative 12240 > (All other decision alternatives)

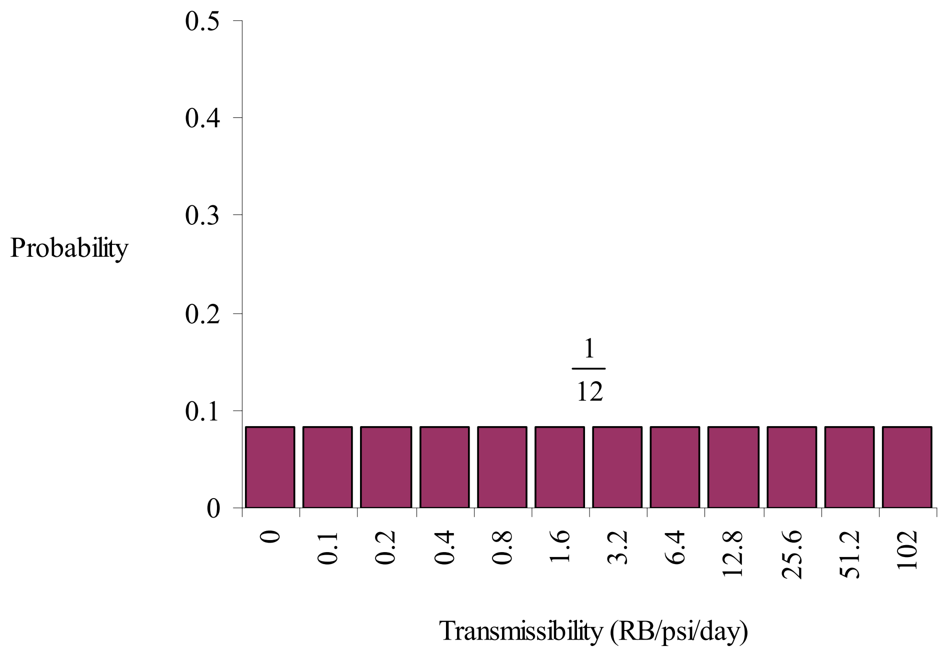
The method for decision-based priors assigns the same probability of ¼ to the four preference outcomes. Other than these four decision alternatives, the remaining 12,237 decision alternatives make no difference to the preference outcomes and non-informative probabilities, because they are not preferred in any particular state of nature.

Because the state S_1 is the only state corresponding to the preference outcome, Alternative 3285>, and the state S_6 is the only state corresponding to the preference outcome, Alternative 12190>, the probabilities for S_1 and S_6 are equal to ¼. For the states of nature, S_2 through S_5, corresponding to the preference outcome, Alternative 3340>, equal probabilities are assigned - there is little difference in the increment of utilities. In the same manner, the states of nature, S_7 through S_{12}, with the same preference outcome, Alternative 12240> have the same probabilities.

The decision-based non-informative prior probabilities are shown in Figure 7.23. The shapes of the non-informative prior probability distributions are different from each other for the decision-based method (Figure 7.23a) and the principle of insufficient reason (Figure 7.23b). The non-informative prior from the decision-based method shows two spikes (larger probabilities) at T=0 and T=1.6 RB/psi/day. Those states of nature are more weighted than other states because they are the extreme states that drive a decision.



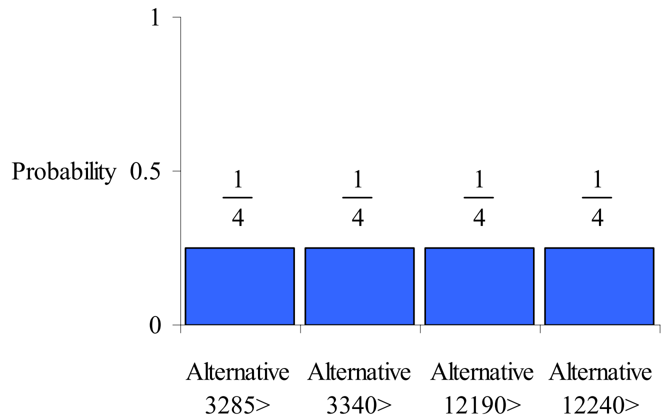
(a) Decision-based method



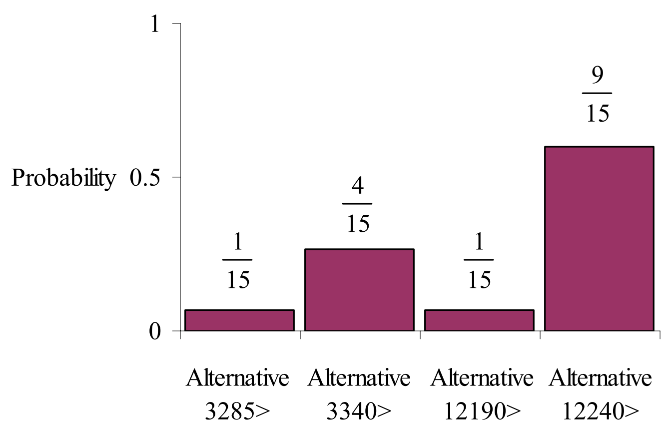
(b) Principle of insufficient reason

Figure 7.23 Non-informative prior probabilities for transmissibility example

The decision-based method provides an unbiased starting point for the decision example shown in Figure 7.24. Four decision alternatives - Alternatives 3285, 3340, 12190, and 12240 - can be the most preferred decision alternative for a given sample space with 12 bins. If the probabilities that each of four decision alternatives is the most preferred are plotted, the decision-based method provides the uniform probabilities shown in Figure 7.24a, while the principle of insufficient reason shows that certain decision alternatives are preferred to others, as shown in Figure 7.24b.



(a) Decision-based method



(b) Principle of insufficient reason

Figure 7.24 Starting point for transmissibility example provided by the decision-based method and the principle of insufficient reason

The principle of insufficient reason and the decision-based method lead to a different decision in this example, as shown in Figure 7.25. The principle of insufficient reason provides that Alternative 12190 (production with three wells from Years 1 through 10 for Unit 1 and no well for Unit 2) is the best alternative, giving an expected monetary value of $16.8 MM. In a decision-based method, Alternative 3340 (production with two wells from Year 1 through 10 for Unit 1 and three wells from Year 1 through 10 for Unit 2) is the best decision alternative, with an expected value of $15.9 MM.

The decision based on the principle of insufficient reason focuses on high return from the oil production. The states of nature with large benefit, such as states, S_{10} through S_{12}, are weighted much more than the other states of nature. The decision based on the principle of insufficient reason yields Alternative 12190, which works profitably for states with large transmissibility but is not economical for states with small transmissibility, as shown in Figure 7.26. However, the method for decision-based priors has no bias in the preference in decision alternatives, so the method chooses Alternative 3340. Alternative 3340 is the plan that places production wells in both units. It gives the net profit that varies little with the states of nature, as shown in Figure 7.26.

The difference in non-informative probabilities may produce the difference in the value of information about T. The value of perfect information analysis provides a VPI of $2.5 MM when the principle of insufficient reason is applied, and $1.9 MM when a decision-based method is employed. Because the principle of insufficient reason leads to Alternative 12190, which produces a large profit only for some states, there is potential risk of a significant loss if the transmissibility is small. The decision-based method, on the other hand, gives Alternative 3340, which works fine with any states of nature. Additional information does not make any contribution on avoiding risk, because the loss is small whatever the transmissibility. This result does not mean that the VPI from the decision-based method is always larger than that from the principle of insufficient reason. This result should not be generalized because the prior probability distribution is no the only factor affecting the VPI. The VPI is also affected by the other factor, a set of consequences in decision matrix.

Alternative 12190

Year	1	2	3	4	5	6	7	8	9	10
Unit 1	3 wells									
Unit 2	No well									

(a) Method based on the principle of insufficient reason

Alternative 3340

Year	1	2	3	4	5	6	7	8	9	10
Unit 1	2 wells									
Unit 2	3 wells									

(b) Decision-based method

Figure 7.25 Different optimal decisions by the principle of insufficient reason and the decision-based method

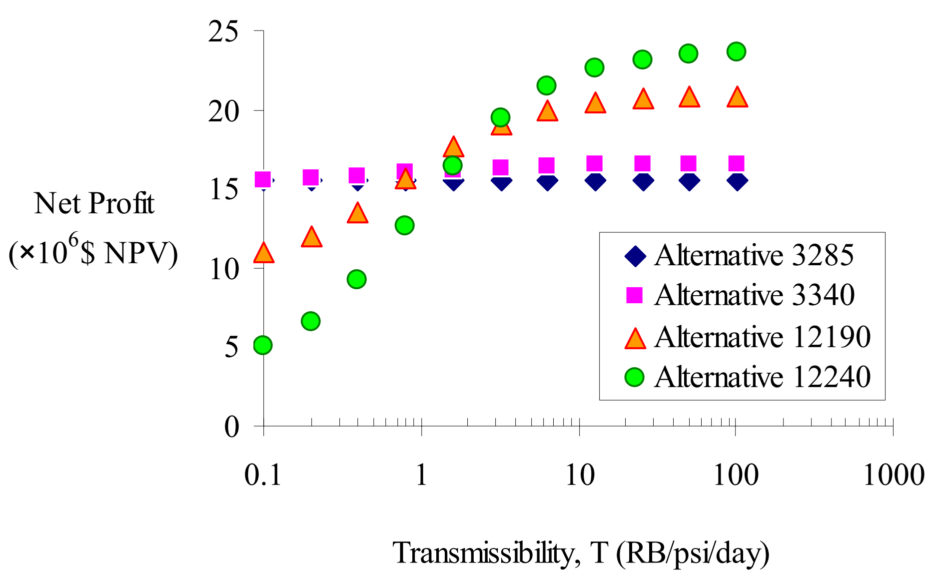


Figure 7.26 Consequences made by each pair of decision alternatives and states of nature.

7.3.7 Non-Informative Prior Probabilities and Decision with Information

Bayes' theorem helps associate new knowledge from information to the previous state of knowledge represented by a non-informative probability distribution. Suppose we have vague information that is described by the likelihood of the information for each state of nature:

P(Information|Transmissibility<25 RB/psi/day)=0.2

P(Information|Transmissibility>25 RB/psi/day)=0.9

The likelihood function means that the information is much more likely when the transmissibility is greater than 25 RB/psi/day and less likely when the transmissibility is less than 25 RB/psi/day. The likelihood function represents a way to illustrate no more than the given information contains. Therefore, it does not rely on mathematical assumptions, such as Gaussian and/or lognormal probability distributions. The likelihood function is shown in Figure 7.27 and the updated probabilities through Bayes' theorem are shown in Figure 7.28. The updated probabilities have more weight on states where the transmissibility is less than 10 RB/psi/day.

Due to the changes in probabilities, decision making with the information produces different expected utilities for the decision alternatives. With the information, the decisions for both the principle of insufficient reason and the method for decision-based priors are the same - Alternative 12190 (three wells for Unit 1 through 10 years and no well for Unit 2), as shown in Figure 7.29.

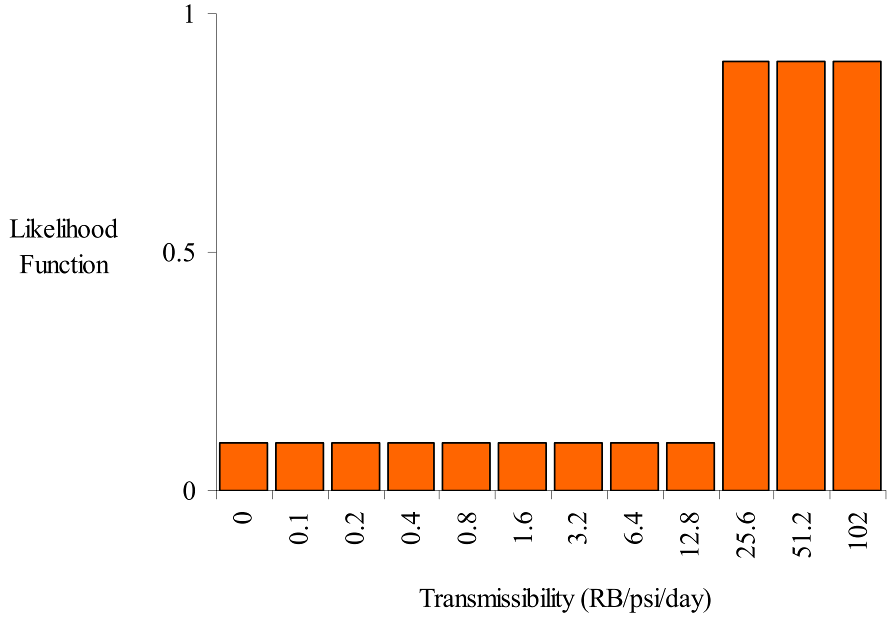


Figure 7.27 Example of a likelihood function, which represents the information by saying that the information is more likely for T<10 (RB/psi/day)

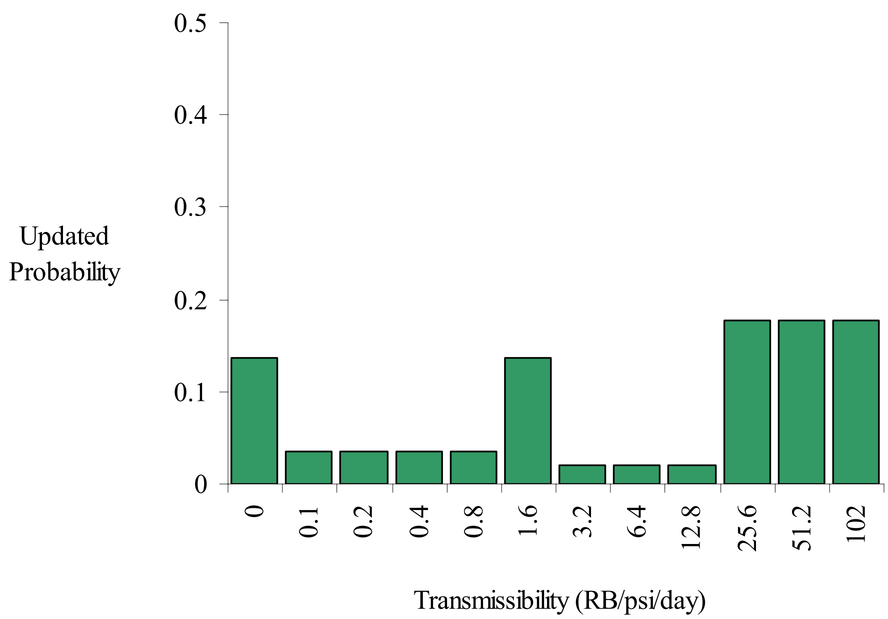


Figure 7.28 Posterior probabilities updated with previous state of knowledge (complete ignorance) in Figure 7.23a and the likelihood function for new information in Figure 7.27.

Alternative 12190

Year	1	2	3	4	5	6	7	8	9	10
Unit 1	3 wells									
Unit 2	No well									

(a) Method based on the principle of insufficient reason

Alternative 12190

Year	1	2	3	4	5	6	7	8	9	10
Unit 1	3 wells									
Unit 2	No well									

(b) Decision-based method

Figure 7.29 Decisions by the principle of insufficient reason and decision-based method with information

Suppose we have opposite information that is still vague, as described by the likelihood of the information for each state of nature:

P(Information|Transmissibility<1 RB/psi/day)=0.9

P(Information|Transmissibility>1 RB/psi/day)=0.2

The likelihood function is shown in Figure 7.30, and the updated probabilities through Bayes' theorem are shown in Figure 7.31. Because the information implies that states with large transmissibility are more likely, the right tail of the non-informative prior probabilities in Figure 7.23 has more weight.

The optimal decision in this case is the same with or without information, as show in Figure 7.25.

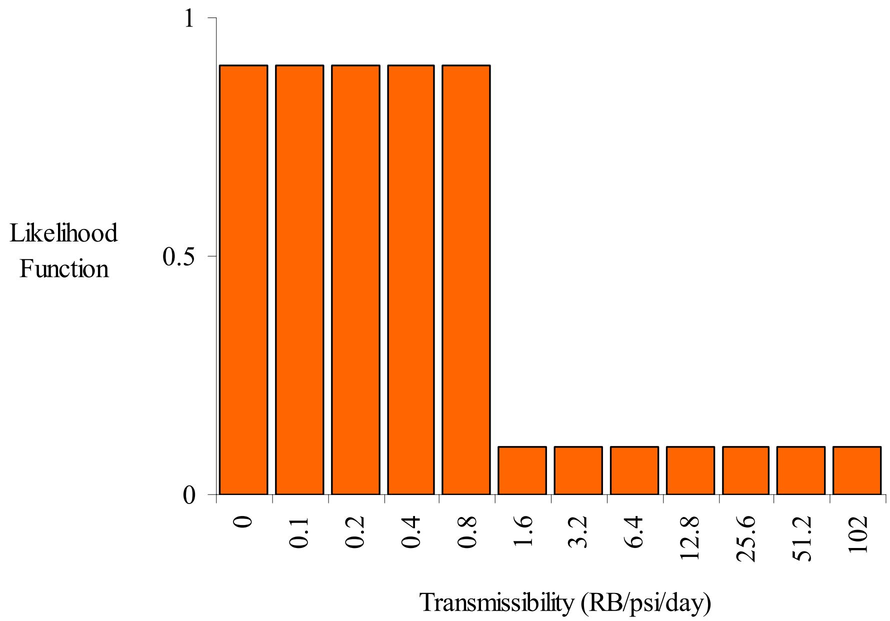


Figure 7.30 Example of a likelihood function, which represents the information saying that the information is more likely for T<1 (RB/psi/day)

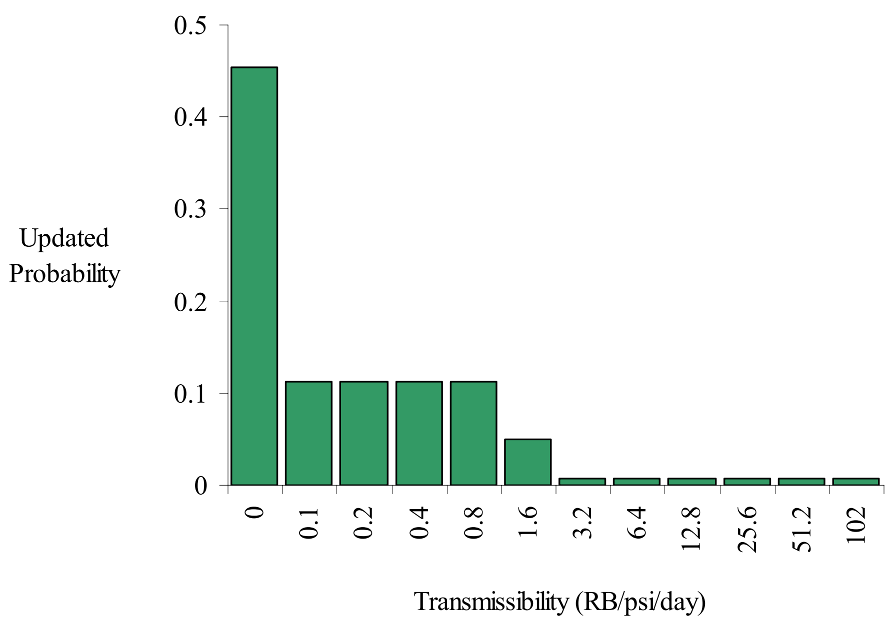


Figure 7.31 Posterior probabilities updated with previous state of knowledge (complete ignorance) in Figure 7.23a and the likelihood function for new information in Figure 7.30.

7.3.8 Non-Informative Prior Probabilities and the Value of Perfect Information

The decision-based method may result in different non-informative prior probabilities when the decision matrix for a decision making problem has been changed. An example of a change in the decision matrix is a change in deterministic variables. In this section, a sensitivity analysis will be conducted with well cost for Unit 1. The response of the sensitivity analysis is the value of perfect information (VPI) in this case. VPI is the maximum value of information that a decision maker can get when the information can clarify with certainty which state of nature will occur. Because VPI depends on the probabilities, the decision-based method is expected to provide a different result from the principle of insufficient reason. The base case of the sensitivity variables is in Table 7.6.

The results of the sensitivity analysis are shown in Figure 7.32. The curves from the principle of insufficient reason and the decision-based method are different from each other. It is not possible to generalize that the VPI of one method is always larger than the VPI of the other. As shown in Figure 7.321, the maximum VPIs occur at different well cost for Unit 1, and the maximum VPIs are not the same.

Detailed information on non-informative prior probabilities, the VPI, and the optimal decision for three values of well cost for Unit 1, is given in Tables 7.10 and 7.11. The results are reasonable in that both methods provide the optimal decision with more wells in Unit 1 and less wells in Unit 2 if the well cost for Unit 1 is small, and with less wells in Unit 1 and more wells in Unit 2 otherwise. While the principle of insufficient reason provides the same uniform probabilities for any well cost for Unit 1, the decision-based method provides different non-information prior probabilities. These different sets of probabilities on states of nature may result in different expected utilities for the decision alternatives, different optimal decision, and different VPIs.

The same optimal decision does not guarantee the same decision-based non-informative prior. For example, Alternative 12199, selected for the range of well costs for Unit 1 from 0 to 1.5 ($ MM), has three different sets of decision-based non-informative prior probabilities, as shown in Table 7.11.

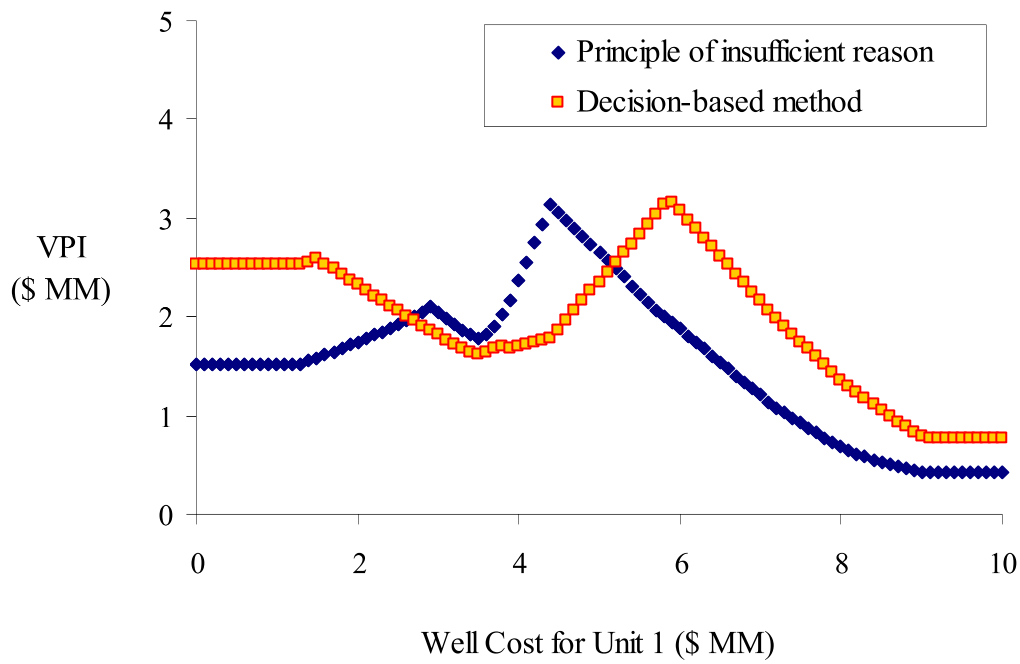


Figure 7.32 Variation of the value of perfect information (VPI) with well cost for Unit 1

Table 7.10 Three values of well cost for Unit 1 and their influence on the decision-based non-informative prior probabilities, the optimal decisions, and the VPIs

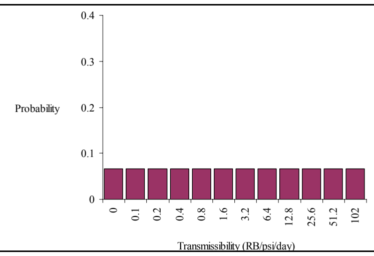
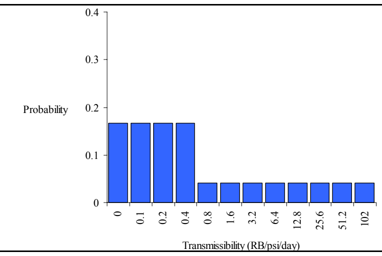
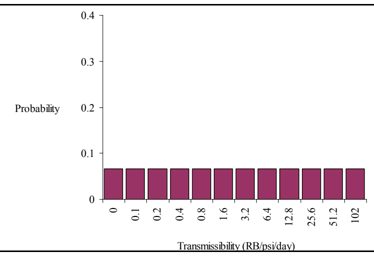
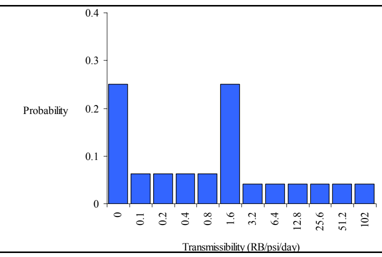
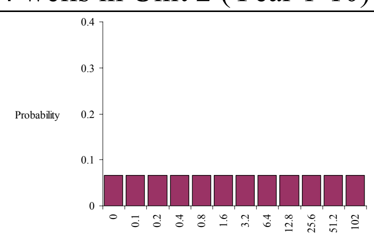
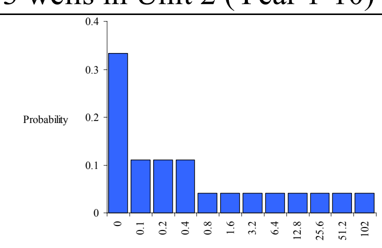
Well Cost for Unit 1		Principle of Insufficient Reason	Decision-Based Method
$1 MM	Non-informative prior probabilities		
	VPI	$1.5 MM	$2.5 MM
	Optimal decision	Alternative 12200 4 wells in Unit 1 (Year 1-10) No well in Unit 2	Alternative 12190 3 wells in Unit 1 (Year 1-10) No well in Unit 2
$4 MM	Non-informative prior probabilities		
	VPI	$2.4 MM	$1.7 MM
	Optimal decision	Alternative 12240 No well in Unit 1 4 wells in Unit 2 (Year 1-10)	Alternative 3340 2 wells in Unit 1 (Year 1-10) 3 wells in Unit 2 (Year 1-10)
$9 MM	Non-informative prior probabilities		
	VPI	$0.44 MM	$0.79 MM
	Optimal decision	Alternative 12240 No well in Unit 1 4 wells in Unit 2 (Year 1-10)	Alternative 12240 No well in Unit 1 4 wells in Unit 2 (Year 1-10)

Table 7.11 The influence of the well cost for Unit 1 on the optimal decisions and the decision-based non-informative prior probabilities

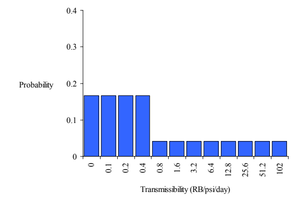
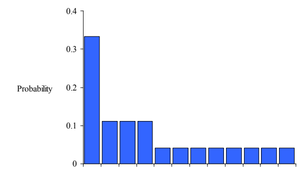
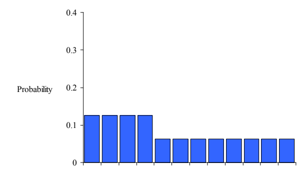
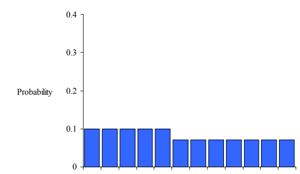
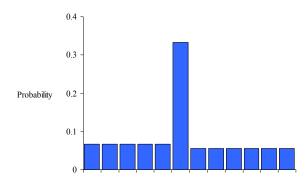
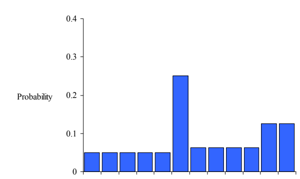
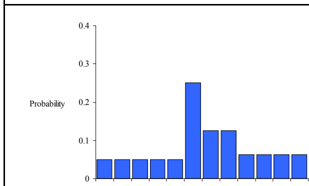
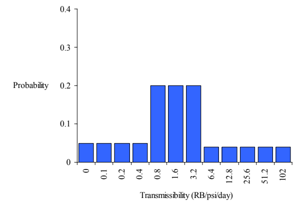
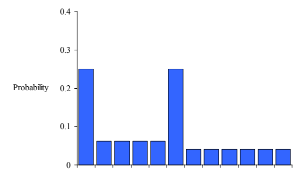
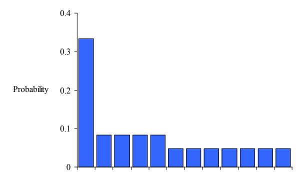
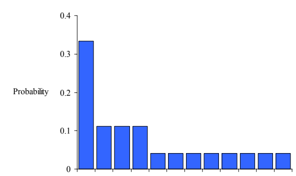
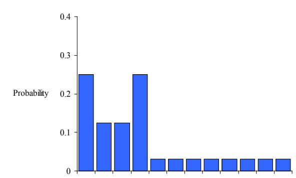
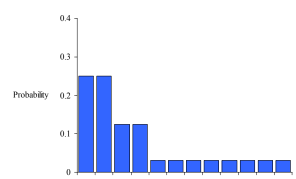
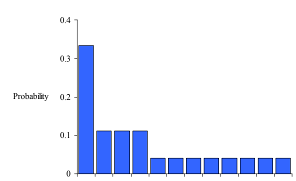
Well Cost for Unit 1 ($ MM)	Optimal Decision	Decision-Based Non-Informative Prior
0-1.3	Alternative 12199 4 wells in Unit 1 (Year 1-9) No well in Unit 2	
1.4		
1.5		
1.6-2.3	Alternative 5005 3 wells in Unit 1 (Year 1-10) 2 wells in Unit 2 (Year 1-10)	
2.4-3.3		
3.4		
3.5		
3.6-3.7		

Table 7.11 (Continued)

Well Cost for Unit 1 ($ MM)	Optimal Decision	Decision-Based Non-Informative Prior
3.8	Alternative 3340 2 wells in Unit 1 (Year 1-10) 3 wells in Unit 2 (Year 1-10)	
3.9-4.4		
4.5-5.6		
5.7-5.8		
5.9-7.1	Alternative 12240 No well in Unit 1 4 wells in Unit 2 (Year 1-10)	
7.2-7.9		
8.0-8.4		
8.5-10		

7.3.9 Summary

The transmissibility example began with a parametric study and history matching of Holstein field to show the importance of heterogeneity associated with decision making in petroleum exploration and production. The transmissibility changed well behavior, profit made by oil production, and ultimately the optimal decision in decision analysis.

The decision-based non-informative prior probabilities for the transmissibility example were given in Figure 7.23a. The decision-based non-informative prior was not uniform, and the detailed discussion on what made the non-informative prior was given.

The method to associate new information was illustrated with exemplary likelihood functions in Figures 7.27 and 7.30. The influence of the new information was highlighted by posterior probabilities and optimal decisions in Figures 7.28, 7.29, and 7.31. It was also shown that non-informative priors are important in decision making if new information is not informative.

A sensitivity analysis was performed to show that different decisions may have different non-informative priors. For the example, a deterministic parameter, well cost for Unit 1, was selected as a variable in the sensitivity analysis. The difference in non-informative priors provided different expected utilities for decision alternatives, different values of perfect information, and different optimal decisions (Figure 7.32 and Table 7.10). The detailed influence of the well cost for Unit 1 was shown in Table 7.11.

7.4 Spatial Variability Example

This spatial variability example is inspired by Journel and Deutsch's work (1993). Journel and Deutsch treat the heterogeneous porous media with no information as a set of stochastic realizations with maximum entropy. In this study, the heterogeneous porous media will be modeled with a multivariate probability density function based on the decision-based method. The decision-based non-informative prior probabilities will demonstrate that the mathematical assumption made for convenience in dealing with complete ignorance, can lead to biased decision making by imposing subjectivity. This will also show the proposed decision-based method provides a basis for decision making without assuming more than the given information.

7.4.1 Objectives

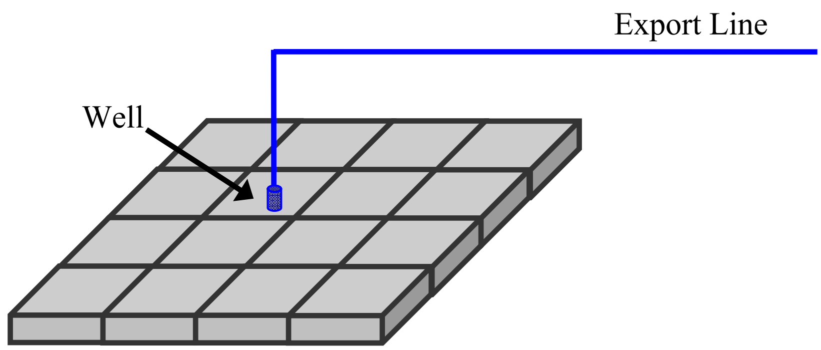
There are three objectives in this spatial variability example. The first is to show the difference in non-informative probabilities given by the principle of insufficient reason and decision-based methods. The second objective is to illustrate the point made by Journel and Deutsch (1993), that applying the principle of insufficient reason to states of nature in a decision making problem is an irrational approach and could lead to under-representing the uncertainty in the outcomes of the decision. Journel and Deutsch indicated that maximizing uncertainty in the input parameters to reflect a lack of information does not guarantee maximum uncertainty in the response of interest. They observed the opposite behavior in their results from geostatistic simulations. The maximum uncertainty model parameters led to minimum variability in response variables, such as breakthrough time and oil recovery. In this spatial variability example, it will be demonstrated that the maximum uncertainty in states of nature may not yield the maximum uncertainty in decision making. The last objective is to show that a change in the decision framework may change the non-informative prior probabilities.

7.4.2 Reservoir Simulator: Two-Dimensional Grid Simulator

A tank model treats a reservoir as a homogeneous unit with infinite transmissibility. The tank-type model is not capable of including spatial variations of petrophysical properties. In actuality, subsurface conditions can have significant spatial variability. For the purpose of considering the spatial variability, multi-dimensional grid simulators are widely used in oil industry. In this study, a simpler type of grid simulator is developed and coded for a decision example.

A schematic diagram of the reservoir for the simulator is shown in Figure 7.33. The reservoir in the model is discretized into 16 cells. The reservoir simulator provides an implicit solution for time histories of reservoir and wellbore pressure at each cell. The time history of the production rate can be easily calculated from the pressure history at the well location and the productivity index.

The flow between adjacent cells is modeled with the transmissibility at each cell face. The transmissibility is equal to a harmonic average of the permeabilities of the cells. Well production is modeled by Darcy’s law, with the productivity index as in the tank model. Material balance for each cell yields 16 equations, where the unknowns are 16 values of pressure at each cell after a certain time interval. In the mathematical sense, this model involves solving 16 simultaneous equations. The pressure time histories can be built by repeating the process above at different time steps.



→

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

Figure 7.33 Schematic diagram for two-dimensional grid reservoir simulator

This model also has two solutions: pressure-constrained and rate-constrained. These constraints are on the production well, not the model. The two cases crate differences in constructing the simultaneous equations, but they make use of the same process of matrix inversion. Figures 7.33 through 7.35 show the results of example cases, one with a constraint on the production rate (Figure 7.34) and the other without the limit (Figures 7.34 and 7.35). The input for both cases is shown in Table 7.12. For both cases, pressure histories at each element are obtained (upper figures in Figures 7.33 and 7.34) and production rate histories are computed based on pressure history at the well location, (2,2) in this example. As seen in Figure 7.35, this reservoir model is capable of simulating transient flow when the production rate is not constrained. The MATLAB code used in production modeling is shown in Appendix D.

Table 7.12 Input for example cases of two-dimensional grid reservoir simulator

Parameter		Pressure-constraint	Rate-constraint
Drainage area, A	(acres)	300	
Porosity, ϕ	(%)	30	
Net pay thickness, h	(ft)	100	
Total compressibility, c_t	(psi^{-1})	0.00005	
Oil viscosity, μ	(cp)	0.8	
Initial reservoir pressure, P_{ini}	(psi)	2,500	
Designated wellbore pressure, P_{wf}	(psi)	2,000	
Radius of wellbore, r_w	(ft)	0.5	
Oil formation volume factor, B_o	(RB/STB)	1.0	
Shape factor, C_A	(-)	30.88	
Skin factor, s	(-)	0	
Time increment, Δt	(day)	5	
Maximum production rate, q_{Lim}	(bbl/day)	∞	800

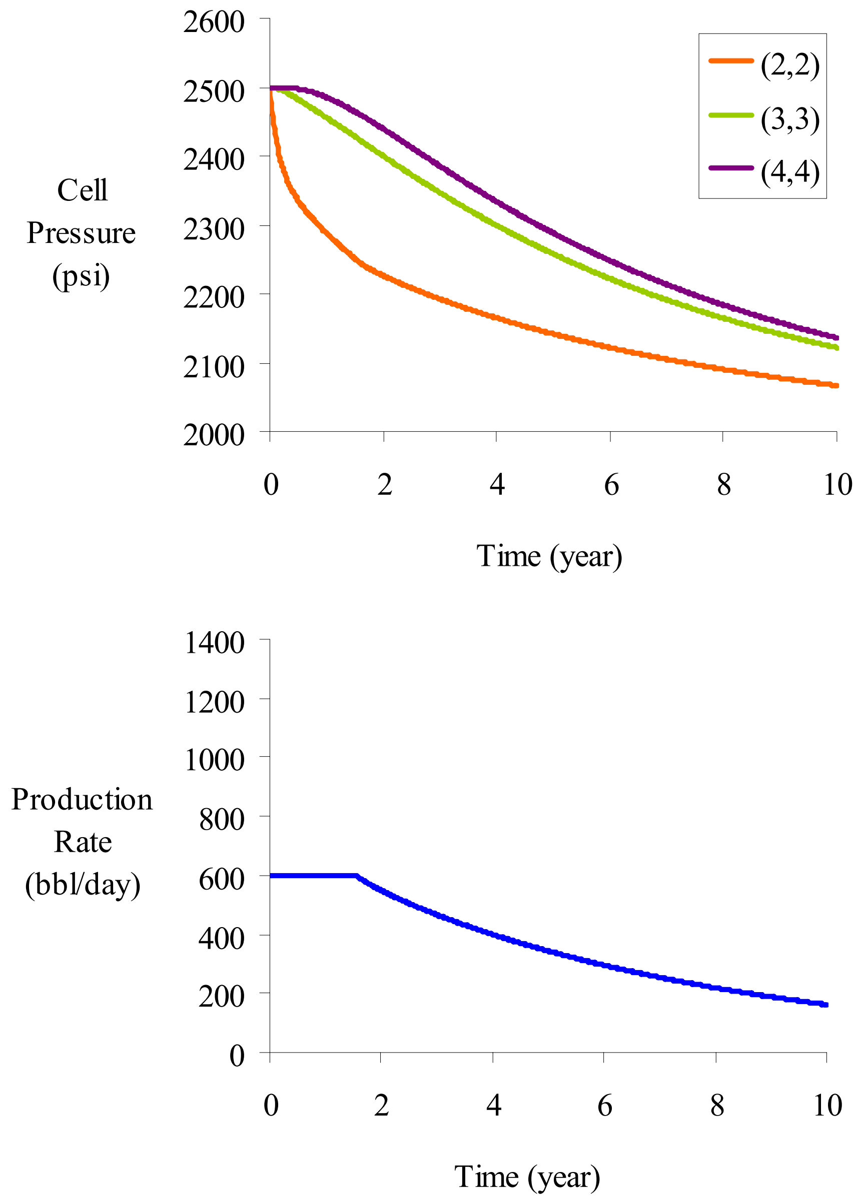
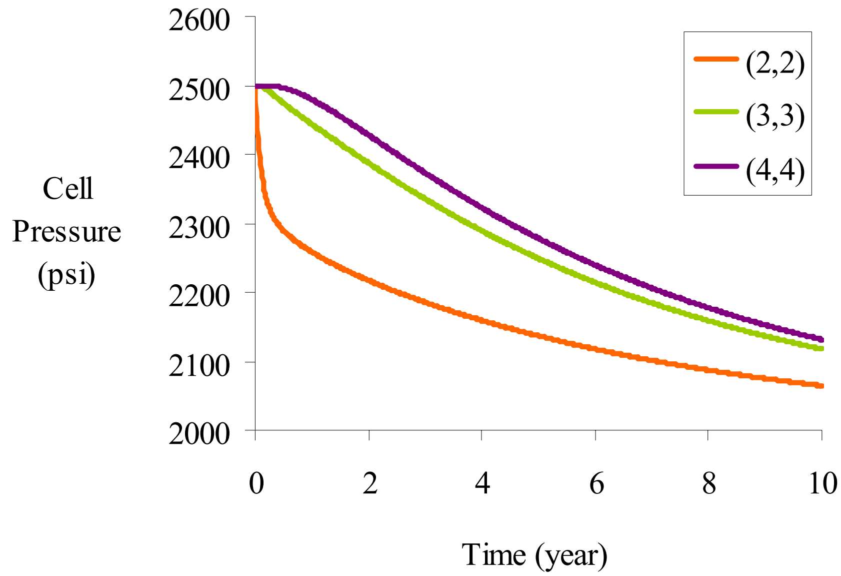


Figure 7.34 Well behavior of the example case under a production rate limit



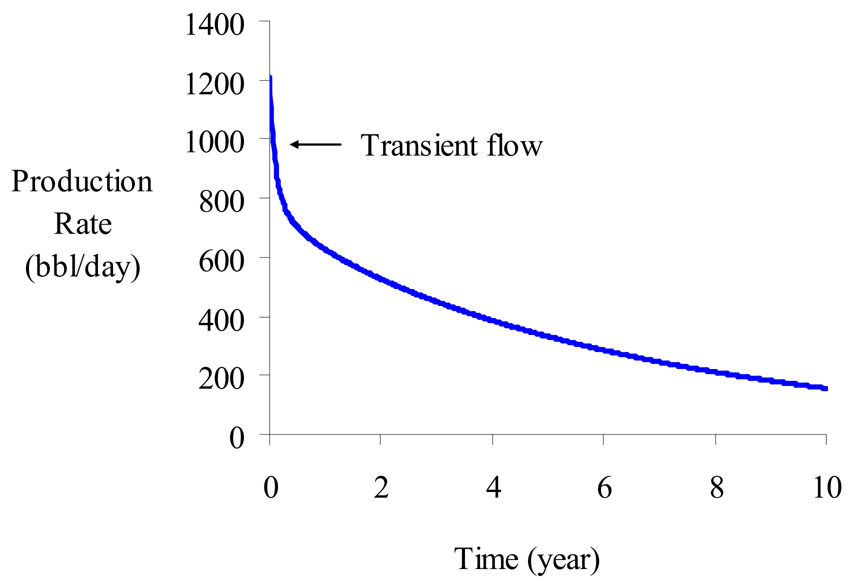


Figure 7.35 Well behavior of the example case under no constraint on production rate

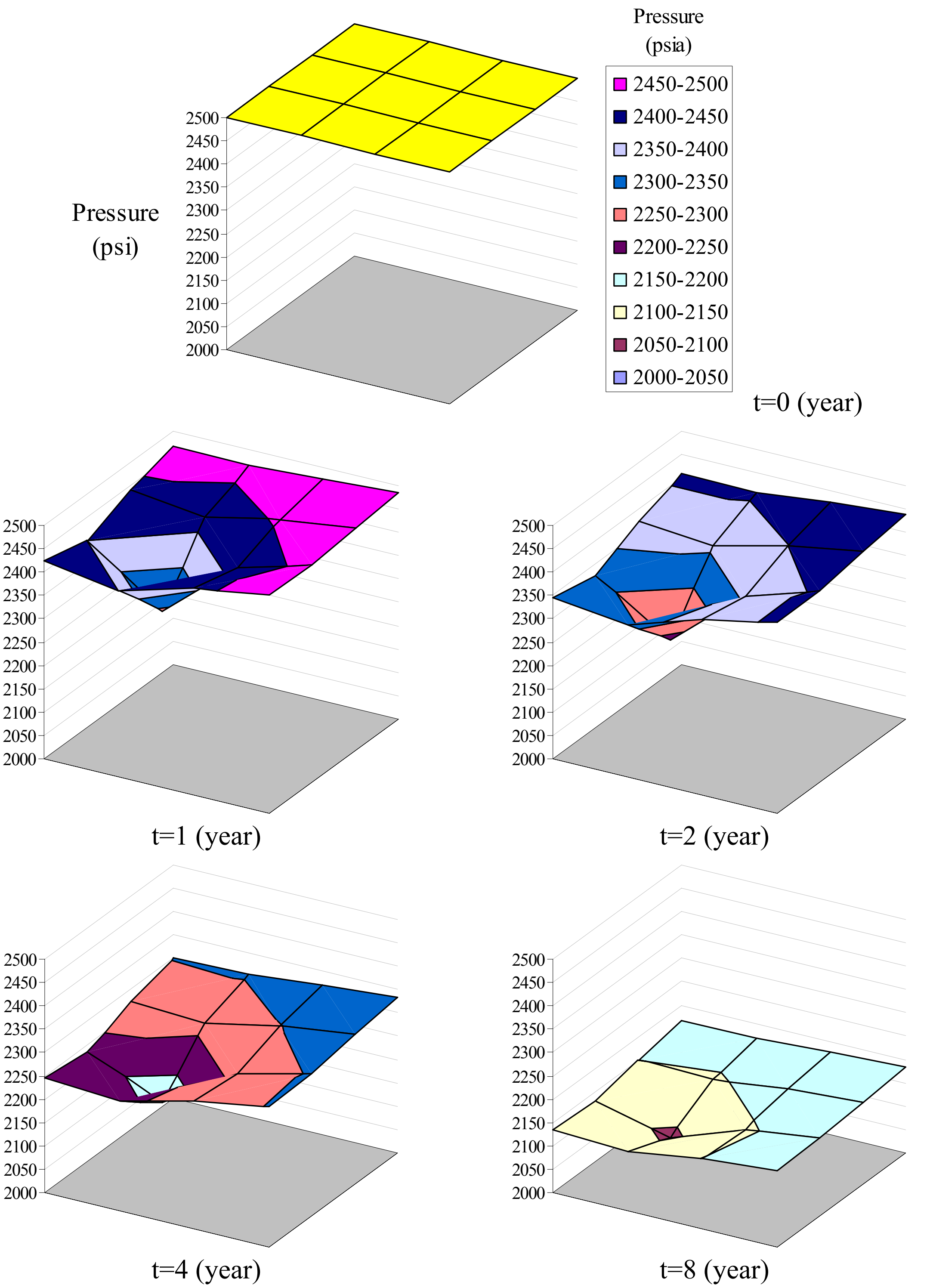


Figure 7.36 Pressure distributions over domain and their decay with time (No constraint on the production rate)

7.4.3 Decision Description

A company wants to make a decision about the number of wells for a 200-acre oil reservoir, as shown in Figure 7.37. The location of wells is near the center of the estimated drainage area, and the maximum number of wells is two, considering the export pipeline capacity and the scale of this project. There are three decision alternatives: abandon this oil field, single well production, and production with two wells.

Most of the physical and economical parameters are deterministic, as presented in Table 7.13. The only uncertain variable is the spatial distribution of permeability, k, which is one of major parameters affecting oil production. The whole drainage area is divided into 16 cells and identification numbers are assigned to every grid for convenience, as shown in Figure 7.38. The uncertain variable can be described with 16 separate uncertain variables - k for cell 1, k for cell 2, and so on. In this example, the multivariate probability distribution is discretized with three possible permeability values - 2, 4, and 8 (md) - for each of the 16 variables. This simplified case yields a large number of states of nature. The bins for the discretized multivariate probability distribution consist of 3^{16} =43,046,721 states of nature.

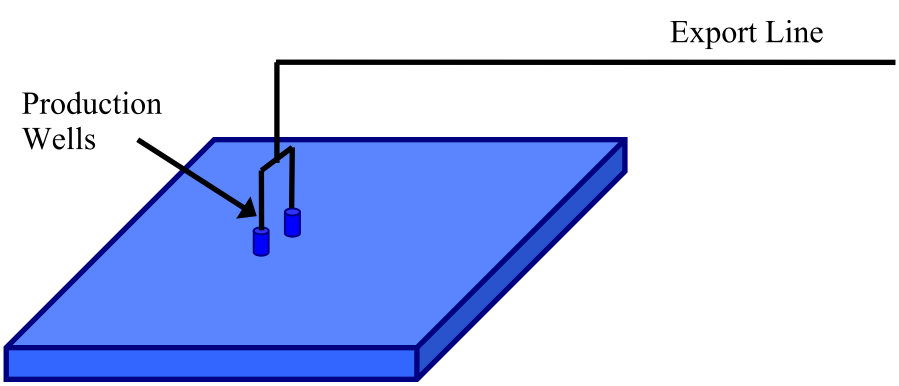


Figure 7.37 Oil reservoir for the spatial variability example

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 7.38 Identification numbers for each grid

Table 7.13 Assumed parameters for the spatial variability example

Parameter		Value
Drainage area, A	(acres)	200
Porosity, ϕ	(%)	30
Net pay thickness, h	(ft)	200
Total compressibility, c_t	(psi^{-1})	0.00005
Oil viscosity, μ	(cp)	0.8
Initial reservoir pressure, P_{ini}	(psi)	2,500
Designated wellbore pressure, P_{wf}	(psi)	2,000
Radius of wellbore, r_w	(ft)	0.5
Oil formation volume factor, B_o	(RB/STB)	1.0
Shape factor, C_A	(-)	30.88
Skin factor, s	(-)	0
Maximum production rate, q_{Lim}	(bbl/day)	800
Discount rate	(%)	5
Oil price	($/bbl)	30
Facility cost	($ MM)	11
Development cost	($ MM/well)	8

7.4.4 Decision Framework

The decision variable in this example is the number of production wells located at Grid 6. There are three decision alternatives: abandon this reservoir (no well), place 1 well, and place 2 wells. Net profit is calculated by subtracting facility and well cost from the profit from oil recovery. Oil recovery is from the simple reservoir simulator described in Section 7.4.2. With these three decision components, the decision tree for this example is shown in Figure 7.39.

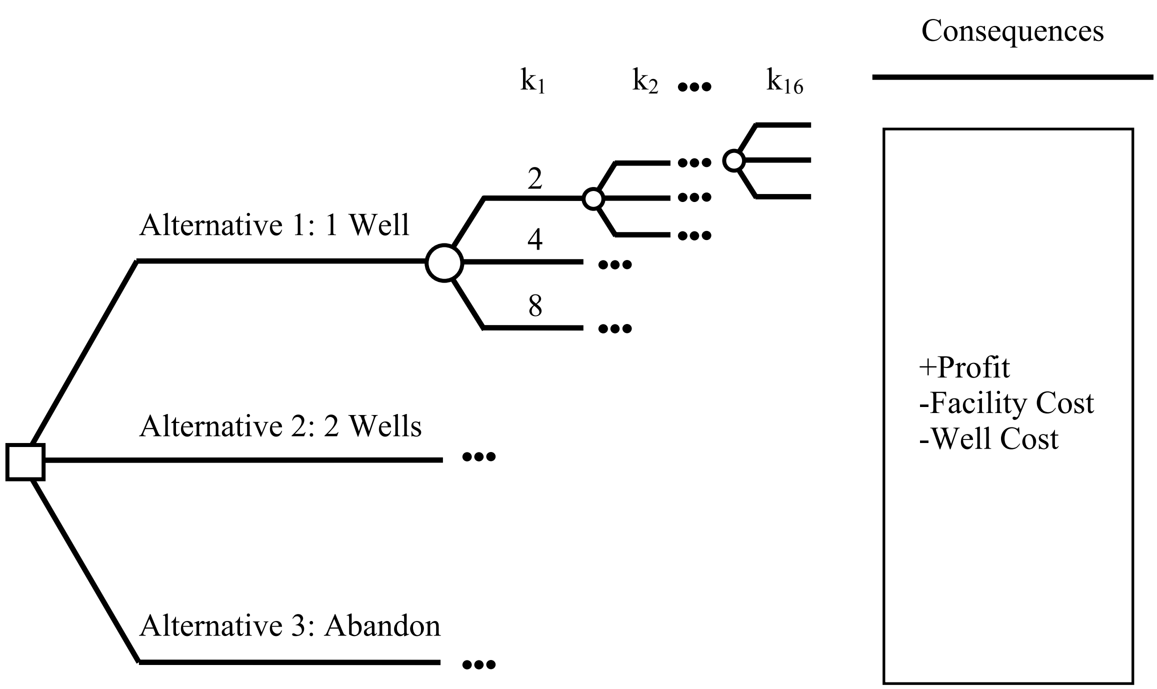


Figure 7.39 Decision tree for the spatial variability example

7.4.5 Non-Informative Prior Probabilities

The spatial variability example has three possible preferred outcomes - $A_1>$, $A_2>$, and $A_3>$ - where A_i represents Alternative *i*. The decision-based non-informative probabilities are summarized with marginal and joint PMFs in Figures 7.39 and 7.40. Comparing the PMFs from the principle of insufficient reason, one can see some of states of nature have more weight by the decision-based method.

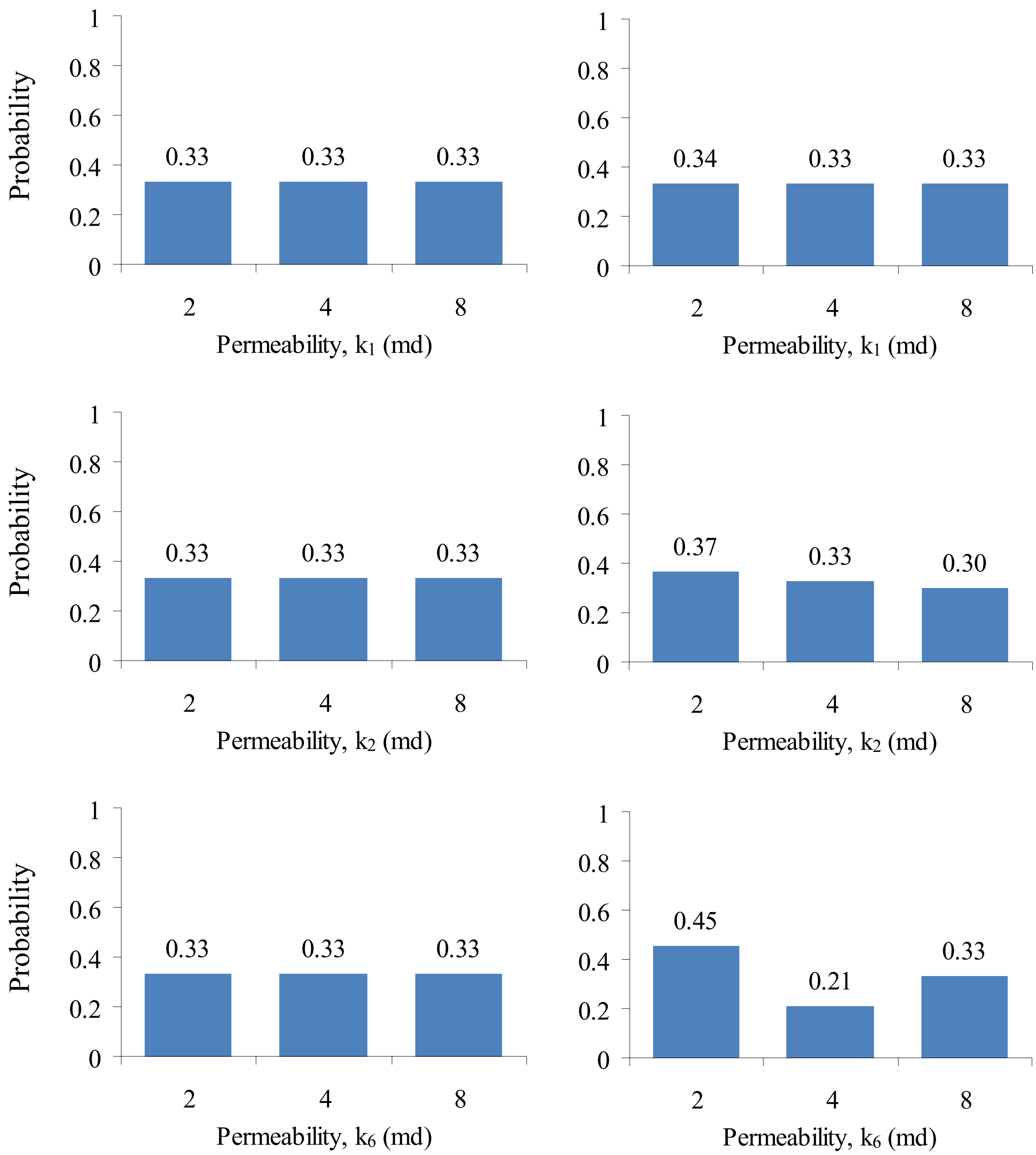
The PMFs in Figures 7.39 and 7.40 illustrate that the relationship between variables is highly non-linear and complicated. The spatial relationship in permeabilities captured by decision-based priors is not Gaussian or lognormal, the probability distributions that are generally assumed when a decision is made in practice.

The reason for the decision-based non-informative prior probabilities can be explained by the distribution of the number of states of nature, as shown in Table 7.14. The event k_6=2 (md) includes two preference outcomes, $A_2>$ and $A_3>$. The contribution of the preference outcome, $A_2>$, on the event, k_6=2 (md), is equal to 8,196,725/(8,196,725+14,348,907)=0.364, because the probability of ⅓ assigned to $A_2>$ is distributed to each possible value of k_6 in proportion to the number of states of nature. In the same manner, the contribution of $A_3>$ on k_6=2 (md) is ⅓. As a result, the decision-based non-informative prior probability of k_6=2 (md) equal to ⅓×(0.364+1)=0.455. The other contribution of $A_2>$, which is equal to (1-0.364) goes to the probability that k_6=4 (md). The non-informative prior probability of k_6=4 (md) is equal to ⅓×(1-0.364)=0.212. Because the preference outcome, $A_1>$, happens only when k_6=8 (md), the non-decision-based non-informative probability that k_6=8 (md) is equal to the probability assigned to the preference outcome, A_1, which is ⅓.

While some pairs of k_6 and preference outcome have dominance of the number of states of nature (Table 7.14), each pair of k_2 and preference outcome has almost the same numbers of states of nature, as shown in Table 7.15. The first preference outcome, A_1>, has the same number of states of nature for three permeabilities - therefore, the contribution of A_1> is the same. The second, A_2>, has fewer states of nature when k_2 is small. The contribution is small when k_2 is small. The opposite happens for A_3>. The contribution of three preference outcomes makes the marginal PMF for k_2, as shown in Figure 7.40.

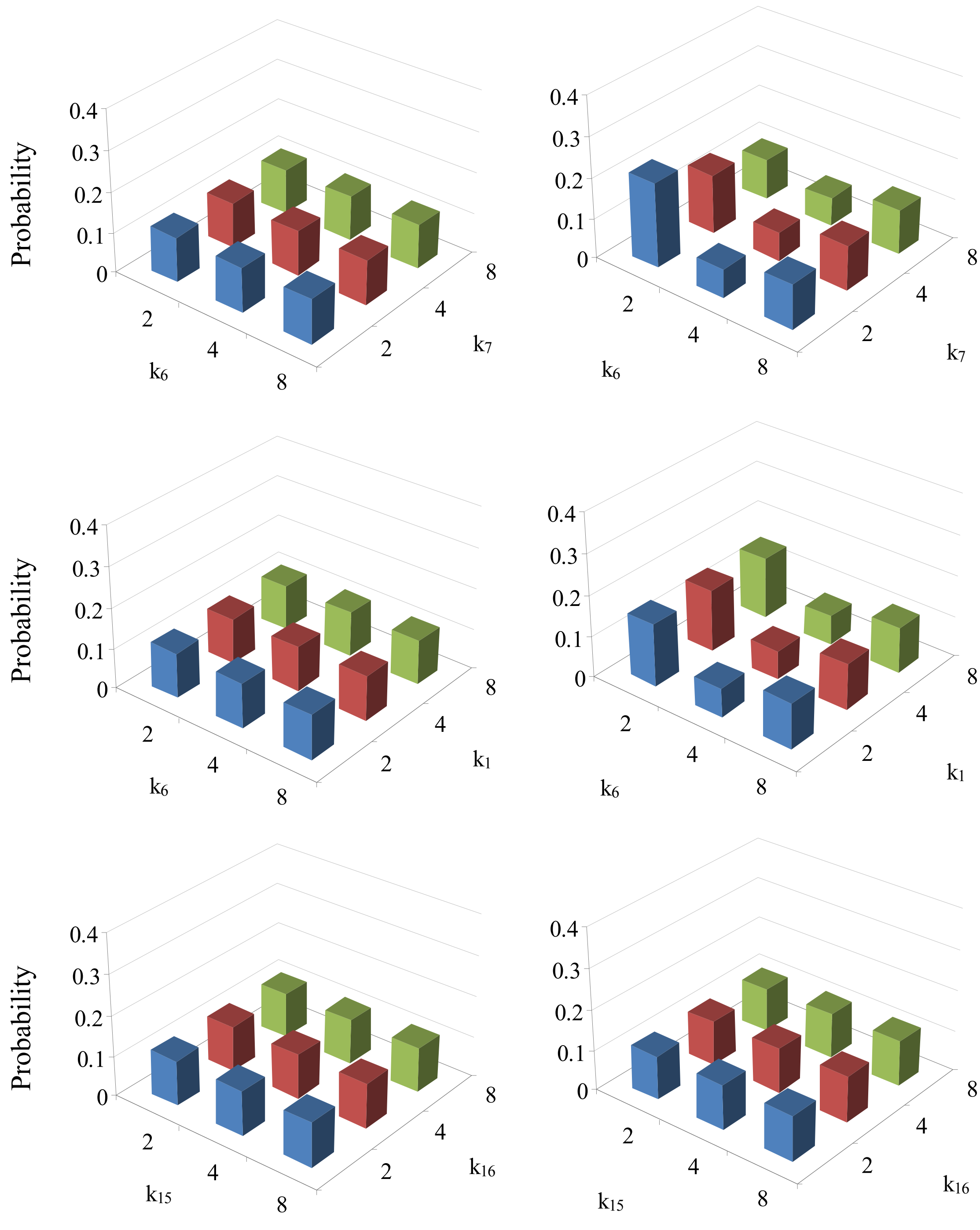
The same algorithm can be used for joint PMFs of decision-based non-informative prior probabilities. The distribution of the number of states of nature is shown in the table at the top of Figure 7.42. The contribution of each preference outcomes is shown in the middle table. The sum of contributions along a row should be equal to unity. The non-informative prior probabilities of each bin is equal to the sum of the product of the probability of a preference outcome (in this case, ⅓, because there are three possible preference outcomes) and contribution of the preference outcome for a given bin. For example, the non-informative prior probability for the first bin, k_6 and k_7 are 2 (md), is equal to $⅓×0+⅓×0.0567+⅓×0.57=0.209$. As seen with this example, the probabilities on a bin of marginal or joint PMFs are affected by the distribution of the preference outcome in the original sample space (states of nature).

The principle of insufficient reason and decision-based methods lead to different non-informative probabilities, different expected utilities for each alternative, and different values of perfect information. However, the optimal decisions were the same for both methods in the example. The expected utilities for both optimal decisions are not the same. These results of decision analysis are summarized in Table 7.16.



(a) The principle of insufficient reason (b) Decision-based method

Figure 7.40 Marginal PMFs of the non-informative prior probability distribution for the spatial variability example



(a) The principle of insufficient reason

(b) Decision-based method

Figure 7.41 Joint PMFs of the non-informative prior probability distribution for the spatial variability example

Table 7.14 The number of states of nature corresponding to each pair of the preference outcome and possible values of the permeability at Grid 6

		k_6 (md)		
		2	4	8
Preference Outcome	A_1>	0	0	14,348,907
	A_2>	8,196,725	14,348,907	0
	A_3>	6,152,182	0	0

Table 7.15 The number of states of nature corresponding to each pair of the preference outcome and possible values of the permeability at Grid 2

		k_2 (md)		
		2	4	8
Preference Outcome	A_1>	4,782,969	4,782,969	4,782,969
	A_2>	6,624,407	7,598,662	8,322,563
	A_3>	2,941,531	1,967,276	1,243,375

The number of corresponding states of nature

	$[k_6,k_7]$ in md								
	[2,2]	[2,4]	[2,8]	[4,2]	[4,4]	[4,8]	[8,2]	[8,4]	[8,8]
A_1>	0	0	0	0	0	0	4,782,969	4,782,969	4,782,969
A_2>	1,277,295	2,913,528	4,005,902	4,782,969	4,782,969	4,782,969	0	0	0
A_3>	3,505,674	1,869,441	777,067	0	0	0	0	0	0

↓

The contribution of each preference outcome on 9 bins

	$[k_6,k_7]$ in md								
	[2,2]	[2,4]	[2,8]	[4,2]	[4,4]	[4,8]	[8,2]	[8,4]	[8,8]
A_1>	0	0	0	0	0	0	⅓	⅓	⅓
A_2>	0.0567	0.1292	0.1777	0.2121	0.2121	0.2121	0	0	0
A_3>	0.5700	0.3039	0.1263	0	0	0	0	0	0

↓

Decision-based non-informative probabilities on 9 bins

	$[k_6,k_7]$ in md								
	[2,2]	[2,4]	[2,8]	[4,2]	[4,4]	[4,8]	[8,2]	[8,4]	[8,8]
Prob.	0.209	0.144	0.101	0.071	0.071	0.071	0.111	0.111	0.111

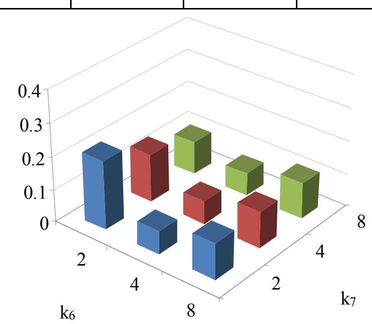


Figure 7.42 Algorithm for establishing decision-based non-informative prior probabilities on the sample space of $[k_6, k_7]$

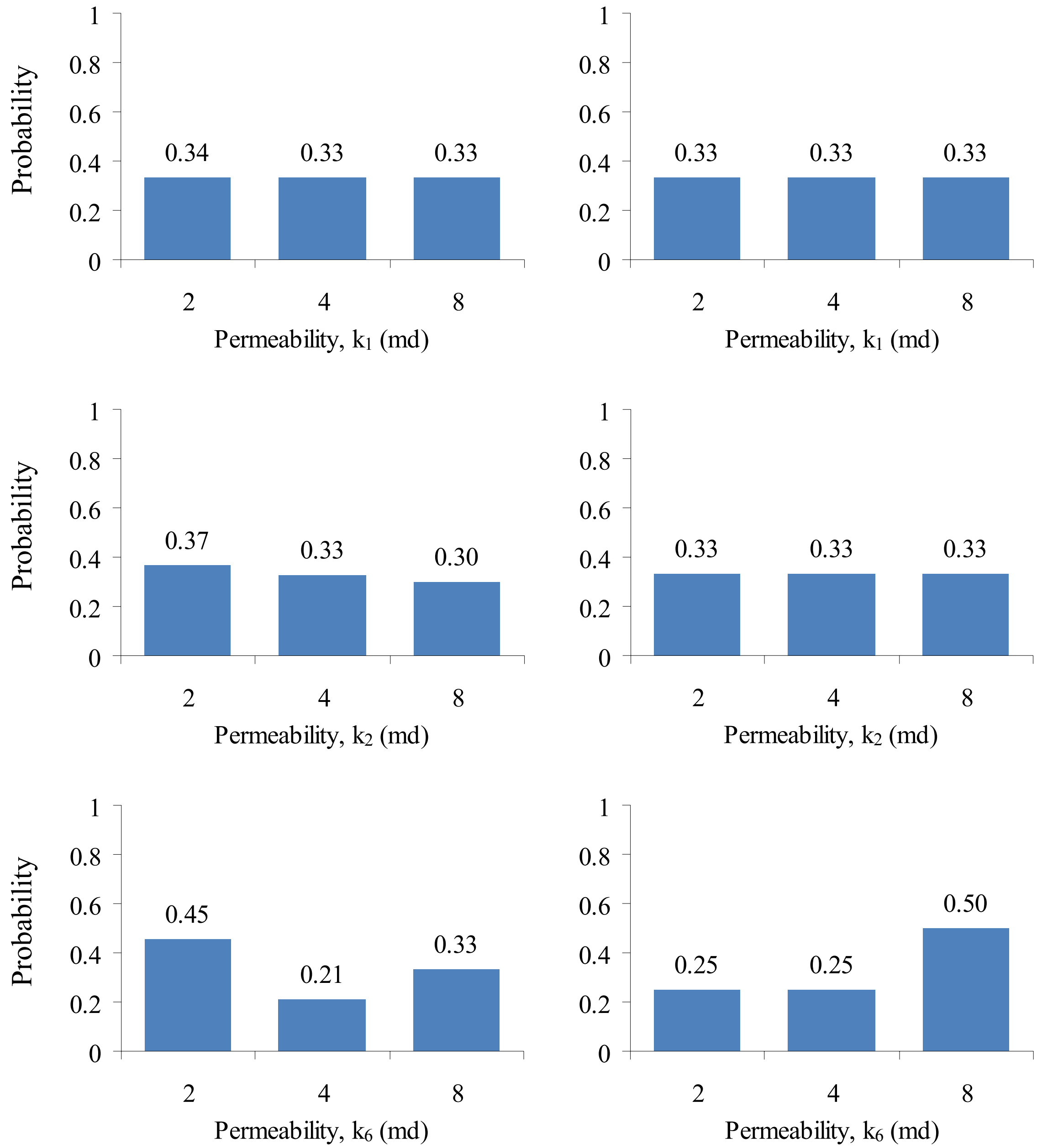
Table 7.16 Decision analysis results

		The Principle of Insufficient Reason	Decision-Based Method
Expected Utility	A_1	\$9,229,036	\$7,888,369
	A_2	\$10,348,773	\$8,818,215
	A_3	\$0	\$0
Optimal Decision		A_2	A_2
Value of Perfect Information		\$419,716	\$514,428

7.4.6 Non-Informative Prior Probabilities from Different Decision Frameworks

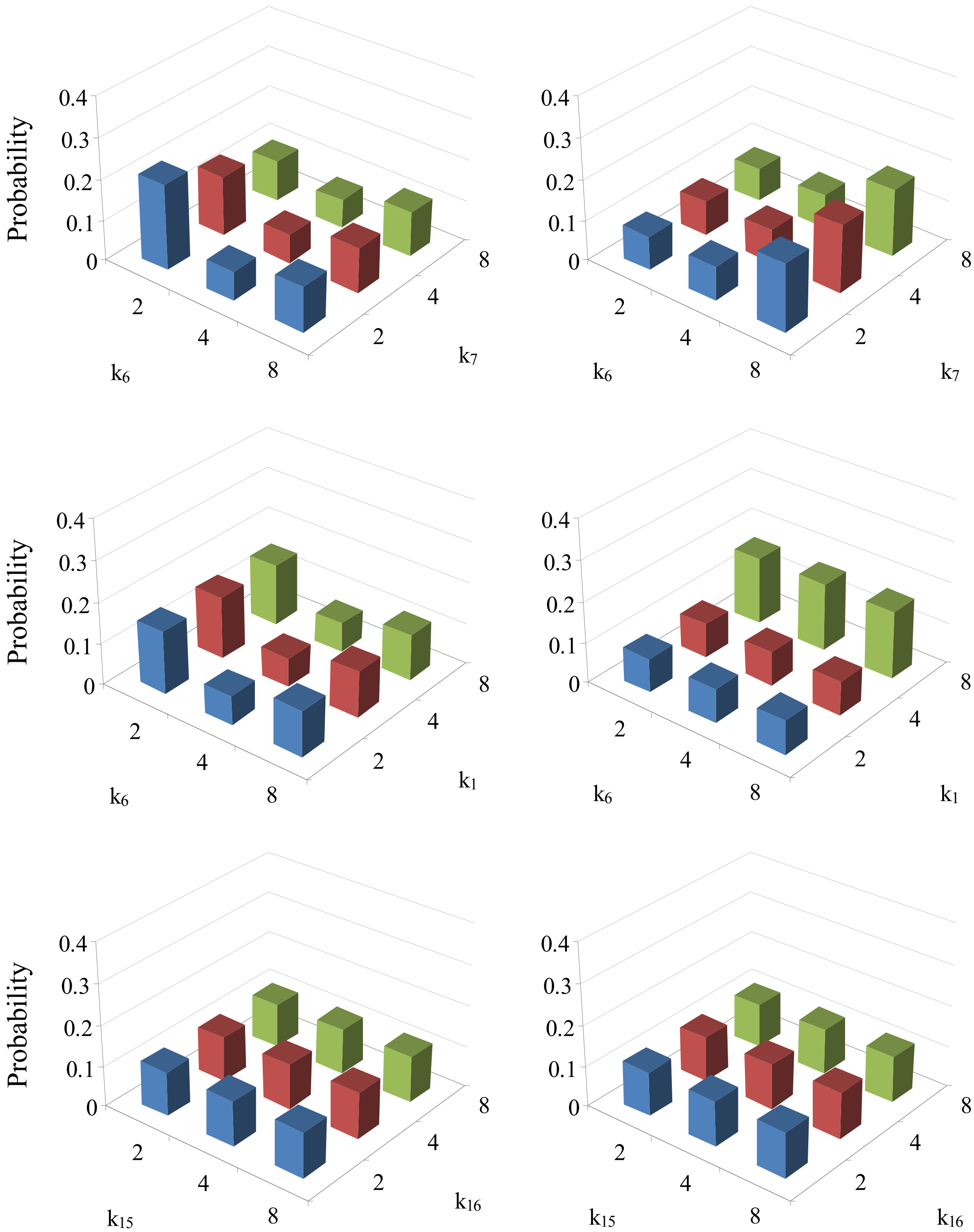
The previous chapter showed that a change in decision framework may change non-informative prior probabilities, even when the states of nature are the same. Possible changes in decision framework include adding or removing decision alternative(s), using different values for deterministic parameters, or using different consequence functions that potentially change preference outcomes by modifying entries in the decision matrix.

If the "No go" decision alternative, A_3, is removed from the set of possible decision alternatives, the possible preference outcomes are A_1> and A_2>. The removal of one decision alternative leads to a large difference in decision-based non-informative prior probabilities. The non-informative prior probabilities for both decision making problems are shown in Figures 7.42 and 7.43.



(a) Three decision alternatives (b) Two decision alternatives

Figure 7.43 Marginal PMFs of the decision-based non-informative prior probabilities for the spatial variability examples with 2 and 3 alternatives



(a) Three decision alternatives (b) Two decision alternatives

Figure 7.44 Joint PMFs of the decision-based non-informative prior probabilities for the spatial variability examples with 2 and 3 decision alternatives

The decision analysis results for the principle of insufficient reason and the decision-based methods are summarized in Table 7.17. Due to different probabilities, both methods provide different expected utilities and values of perfect information for the two decision alternatives. The optimal decision is the same, but the same optimal decision is not able to be generalized. Comparing this result with that for three decision alternatives in Table 7.17, the decision-based method yields different expected utilities and VPIs while the principle of insufficient reason yields only different VPIs. Because the principle of insufficient reason provides the same probabilities for the three- and two-decision-alternative cases, the expected utilities should be the same. However, VPIs may be different because of the way VPI is quantified. VPI is equal to the sum of $P(S_i)\times\{Max(Utility(A_1|S_i), Utility(A_2|S_i), Utility(A_3|S_i))-Utility(A_2|S_i)\}$ (i=1,…,the total number of states of nature) for decision making with three alternatives, and of $P(S_i)\times Max(Utility(A_1|S_i), Utility(A_2|S_i))-Utility(A_2|S_i)$ for decision making with two alternatives. If there exists a state of nature that has A_3 as the most preferred decision alternative, the value for Max() may be different for two decision frameworks and VPI may differ from each other. In addition, if A_3 is always the least preferred for all states of nature, the VPI when based on the principle of insufficient reason should be the same for both decision frameworks.

Table 7.17 Decision analysis results for the spatial variability example with two decision alternatives

		The Principle of Insufficient Reason	Decision-Based Method
Expected Utility	A_1	\$9,229,036	\$12,036,102
	A_2	\$10,348,773	\$12,614,435
Optimal Decision		A_2	A_2
Value of Perfect Information		\$348,628	\$522,941

7.4.7 Summary

The spatial variability example showed the difference in non-informative probabilities given by the principle of insufficient reason and decision-based methods in Figures 7.40 and 7.41. The marginal and joint PMFs from the decision-based non-informative prior were not uniform. This means that the uniform probability distribution based on the principle of insufficient reason may produce an unbiased starting point for decision making. In other words, maximum uncertainty in states of nature may not provide maximum uncertainty in decision outcomes. This example supports the point made by Journel and Deutsch (1993).

It was shown in Figures 7.41 and 7.42 that different sets of decision alternatives may have different decision-based non-informative priors.

Chapter 8. Conclusions and Recommendation

This dissertation proposed a new method to establish a non-informative prior probability distribution for decision analysis. The principle of insufficient reason for the probability distribution under complete ignorance was reviewed and discussed, highlighting three difficulties: bias in decision alternatives, inconsistency, and theoretical unsoundness. The next step was to introduce a decision-based method and discuss the logic and rationality in the decision-based method. This step supports the hypothesis that the decision-based method provides a rational, consistent, and theoretically sound non-informative prior probability distribution. Three decision making examples in engineering practice demonstrated the practical implications of the method for decision-based priors.

8.1 Logical and Rational Basis of Decision-Based method

The decision-based method provides a rational starting point for decision making. The principle of insufficient reason assigns the same probabilities to each state of nature on the basis of randomness in states of nature. However, under complete ignorance a decision maker behaves as if the decision outcomes are random. The decision-based method is based on the concept of random choice and balances decision outcomes that include both of the preference in decision alternatives and the decision consequences.

The decision-based method provides consistent non-informative prior probabilities, while the principle of insufficient reason fails to provide a consistent probability distribution. This inconsistency has been a major issue in defining prior probability distributions for Bayes' theorem, but there has been no unique answer for it. The decision-based method provides an objective method to define the sample space where the principle of insufficient reason should be applied to. The resultant non-informative probabilities are uniquely established for a given decision framework.

The decision-based method provides a theoretical satisfaction to decision theory. There are several axioms in decision theory made from the economic behavior of a rational person. While the principle of insufficient reason fails to satisfy the axioms regarding decision making under complete ignorance, the decision-based method works successfully.

8.2 PRACTICAL IMPLICATIONS OF DECISION-BASED METHOD

The decision-based method provides an objective way to represent the uncertainty for decision making under complete ignorance. The process is deterministic because decision makers do not need to provide subjective information on probabilities. The method is applicable to both continuous and discrete random variables.

Decision-based non-informative prior probabilities capture a highly non-linear and complex relationship between input variables. In some decision making problems, the decision-based method weights the extreme state of nature that may be driving a decision.

Non-informative probabilities for a decision making problem may be different from those for another decision making with the same set of states of nature. The reason for different non-informative probabilities is that the two decision making problems may have different preference outcomes that the decision-based method is based on. This difference affects the expected utilities for decision alternatives, optimal decisions, and value of perfect information.

The decision-based method can be applied with the algorithm provided in this study. The algorithm is straightforward and requires only a decision matrix as an input.

The computer coded algorithm of the decision-based method is also provided in this study. The algorithm is capable of working for practical decision making with a number of decision alternatives and states of nature as illustrated with practical decision examples.

8.3 RECOMMENDATION FOR FUTURE WORK

It is recommended that rational likelihood functions should be developed because the rational decision making requires not only a rational starting point for decision making, but also a rational likelihood function in Bayesian decision analysis. The realistic and reasonable likelihood function is made from and calibrated with the information, for example, field investigation, laboratory test, and experts' opinion. Because the information is vague and from various sources, to establish a reasonable method to build likelihood functions is challenging. The likelihood functions should include no more than is actually contained in the information. Therefore, the arbitrary use of mathematically convenient probability density functions may under-represent the uncertainty involved in decision making. The likelihood functions in Section 7.3.7 would be a good example for the least biased interpretation of information. The calibrated likelihood functions would provide a rational way to assess the value of information and the value of perfect information.

Appendices

APPENDIX A. HISTORY MATCHING FOR HOLSTEIN OIL FIELD

History matching was performed for the oil production from the BP-operated Holstein field. The goal of the history matching was to estimate model flow parameters from the production history provided by the Minerals Management Service (2008). The model parameters, such as productivity index, decay constant, the difference between initial reservoir pressure and wellbore pressure, and transmissibility between two reservoir units, are for the simple reservoir simulator described in Section 7.3.2. The optimal match was found by minimizing the sum of least squared error. The optimization was performed by Microsoft Excel Solver.

There are twelve production wells in the Holstein field, as shown in Figure A.1. The first task was to select the wells to be used in history matching. For the selection purpose, the history matching was performed with individual wells and the results are shown in Table A.1. Based on the result and the location of wells, the wells are grouped into 4, as shown in Table A.2. Groups 2, 3, and 4 were eliminated from detailed history matching with the modified tank-model. The wells in the middle latitude, A007, A009, A011, and A015, (Group 2) are excluded because they have very low productivity indices. Group 3 has the wells in the lower latitude, A010 and A012, located far from the others and show large productivity indices. A004 in Group 4 was not included because the production period was short and the productivity index is smaller that the others. The history matching was conducted with the rest of production wells in Group 1, A002, A003, A005, A006, and A013.

Table A.1 Summary of history matching results for individual production wells

	Well ID	Productivity Index (bbl/psi/day)	Drainage Area (acres)	Initial Average Reservoir Pressure (psi)
Group 1	A002	5.2	312	4,358
	A003	6.9	116	3,883
	A005	4.3	77	5,539
	A006	5.5	81	5,162
	A013	3.8	157	4,678
Group 2	A007	2.9	46	3,176
	A009	2.2	18	2,304
	A011	1.1	2	6,236
	A015	0.3	200	8,970
Group 3	A010	11.8	200	4,550
	A012	7.1	200	3,949
Group 4	A004	0.1	200	41,880

Table A.2 Summary of history matching results for categorized production wells

Group	Well ID	Productivity Index (bbl/psi/day)	Drainage Area (acres)	Initial Average Reservoir Pressure (psi)
Group 1	A002, A003, A005, A006, A013	3.8-6.9	81-312	3,883-5,639
Group 2	A007, A009, A011, A015	0.3-2.9	46	2,304-8,970
Group 3	A010, A012	7.1, 11.8	200	3,949, 4,550
Group 4	A004	0.1	200	8,970

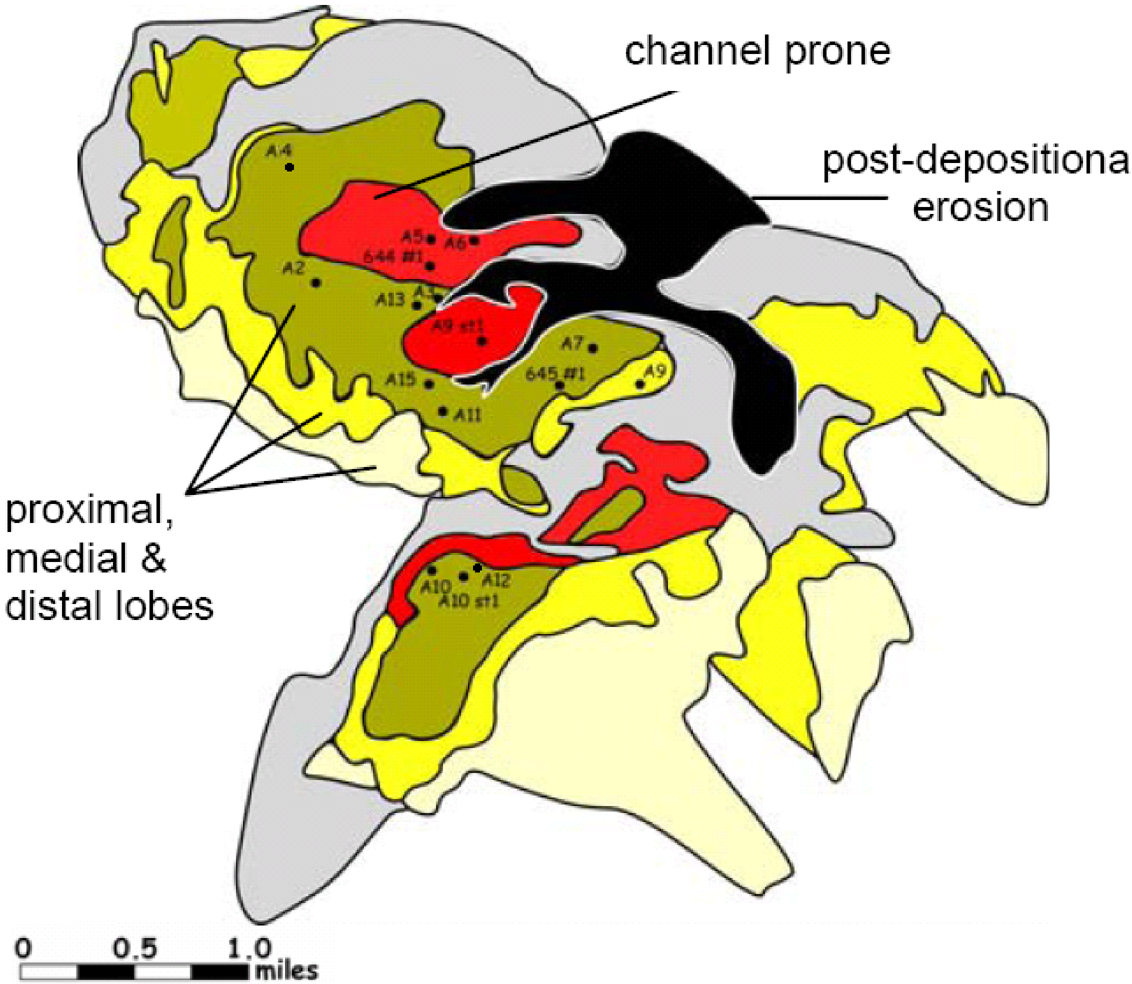


Figure A.1 Reservoir depositional elements map (Wiseman *et al.*, 2007)

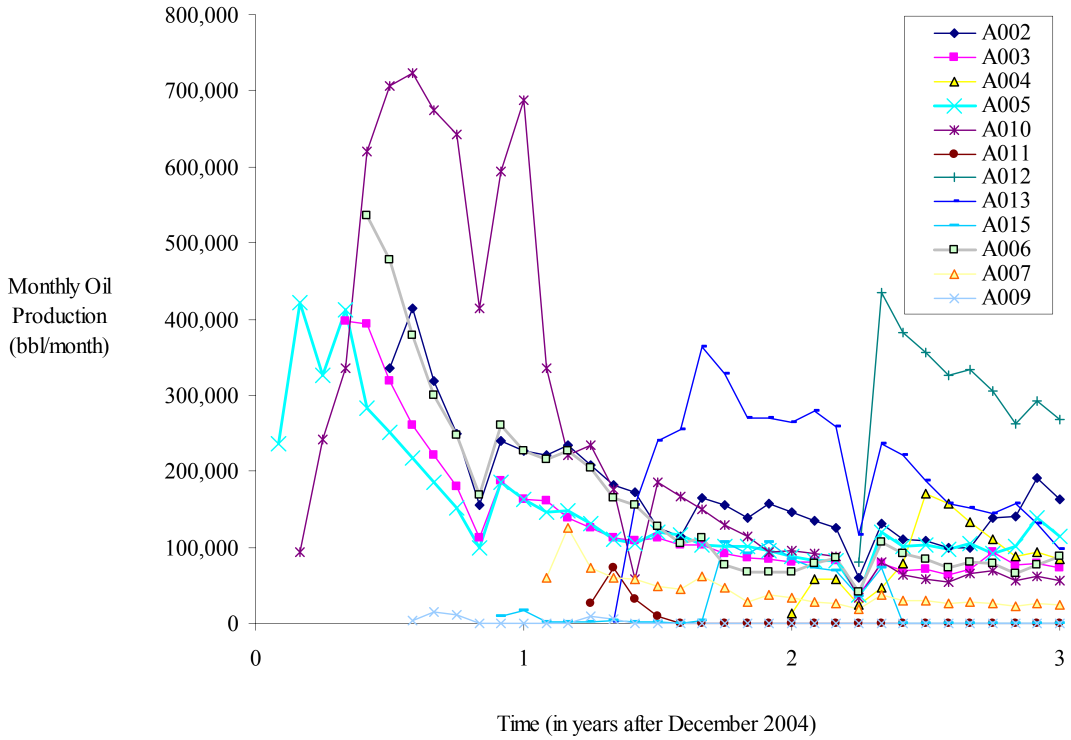


Figure A.2 Oil production history for each well in Holstein field

History matching was performed with the modified version of Tank model. The model is able to simulate the pressure and production rate histories for two reservoir units with communication. Because the production model is limited to two units, the history matching was performed for each pair combination from the five production wells. This approach reduces the degree of freedom, but produces a different set of model parameters for each pair. For example, the productivity index for A002 is 12 (bbl/psi/day) when coupled with A005 but 14 (bbl/psi/day) with A013. This non-uniqueness can be observed in the other pairs in Table A.3. The oil production curves from the MMS database and the optimal match are shown in Figures A.3 through A.10.

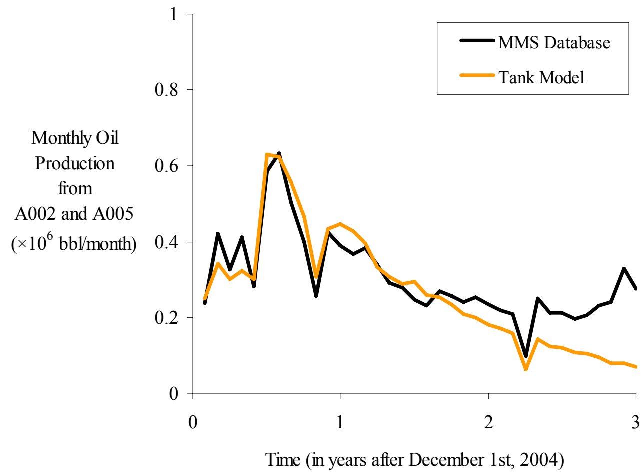
Table A.3 Summary of history matching results for coupled production wells

Well ID	Pair	Productivity Index, J (bbl/psi/day)	Drainage Area, A (acres)	Transmissibility, T (bbl/psi/day)	Initial Pressure Difference, ΔP_{ini} (psi)	
A002	With A005	14	55	7×10^{7}	3085	
	With A013	12	63	0.6	3100	
A005	With A002	11	68	7×10^{7}	3085	
	With A013	18	84	100	2809	partly-matched
	With A003	26	45	60	2575	
	With A006	26	183	60	2572	
A013	With A002	10	100	0.6	3100	
	With A005	25	100	100	2809	partly-matched
	With A006	8	100	3	3338	
	With A003	8	97	3	3375	
A003	With A005	11	28	60	2575	
	With A013	24	187	3	3375	
	With A006	13	137	25	3650	
A006	With A005	49	41	60	2572	
	With A013	15	37	3	3338	
	With A003	14	60	25	3650	

A002

&

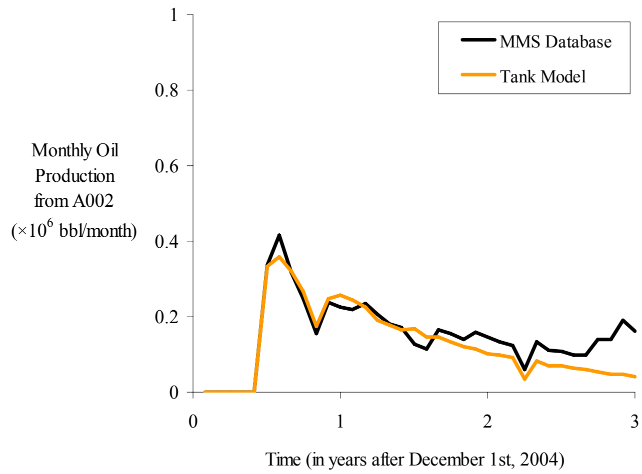
A005



ΔP_{ini}=3085 (psi)

T=7×10^7 (bbl/psi/day)

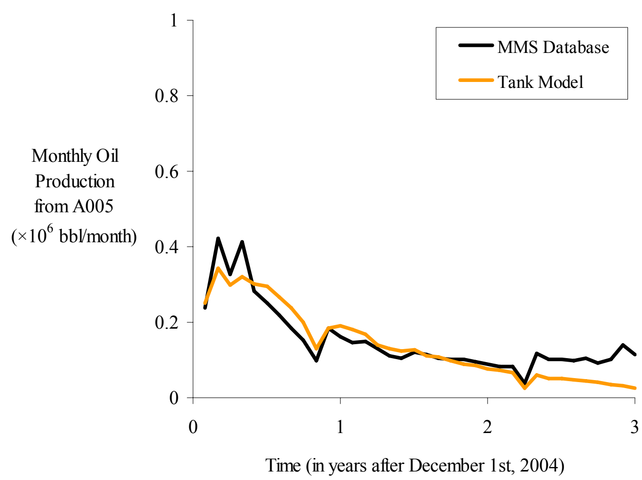
A002



J=14.3 (bbl/psi/day)

A=55.3 (acres)

A005



J=10.6 (bbl/psi/day)

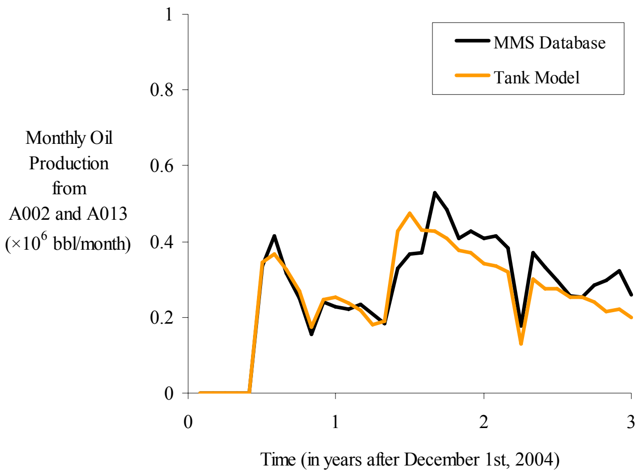
A=68.3 (acres)

Figure A.3 History matching results: A002 and A005

A002

&

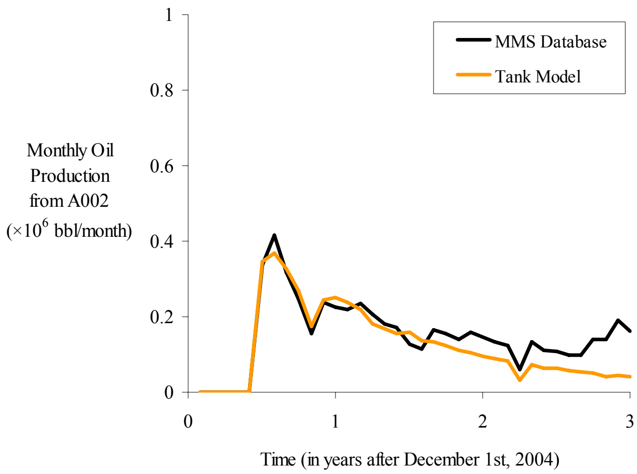
A013



ΔP_{ini}=3101 (psi)

T=0.6 (bbl/psi/day)

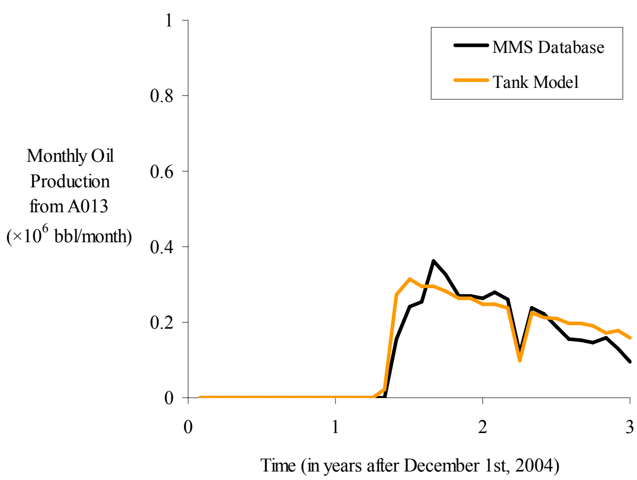
A002



J=12.5 (bbl/psi/day)

A=63.5 (acres)

A013



J=9.8 (bbl/psi/day)

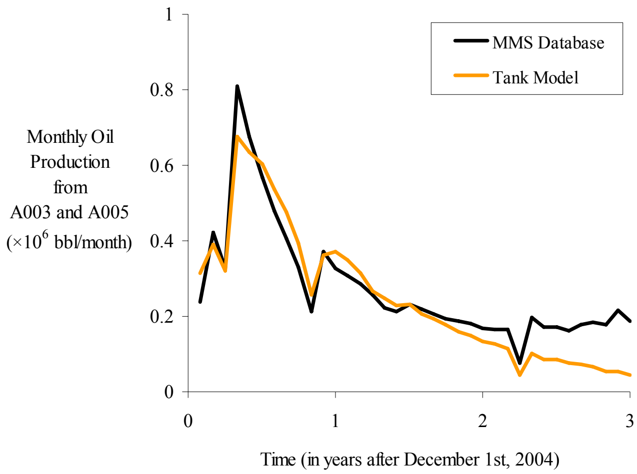
A=100 (acres)

Figure A.4 History matching results: A002 and A013

A003

&

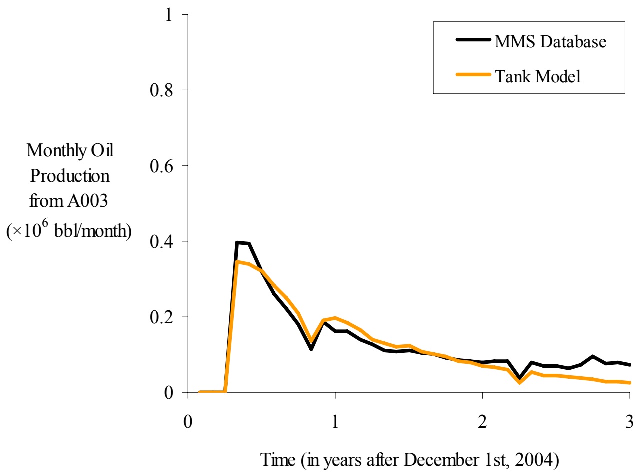
A005



ΔP_{ini}=2575 (psi)

T=60 (bbl/psi/day)

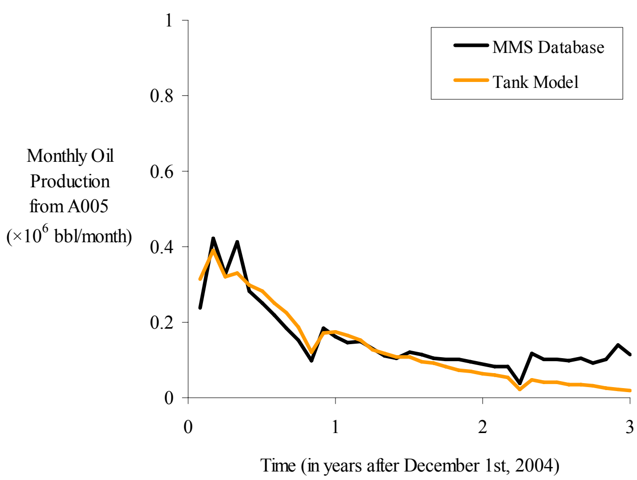
A003



J=23.7 (bbl/psi/day)

A=187 (acres)

A005



J=26.4 (bbl/psi/day)

A=44.7 (acres)

Figure A.5 History matching results: A003 and A005

A003 & A006

ΔP_{ini}=2651 (psi)

T=25 (bbl/psi/day)

A003

J=13.1 (bbl/psi/day)

A=1.37 (acres)

A006

J=14.2 (bbl/psi/day)

A=60.3 (acres)

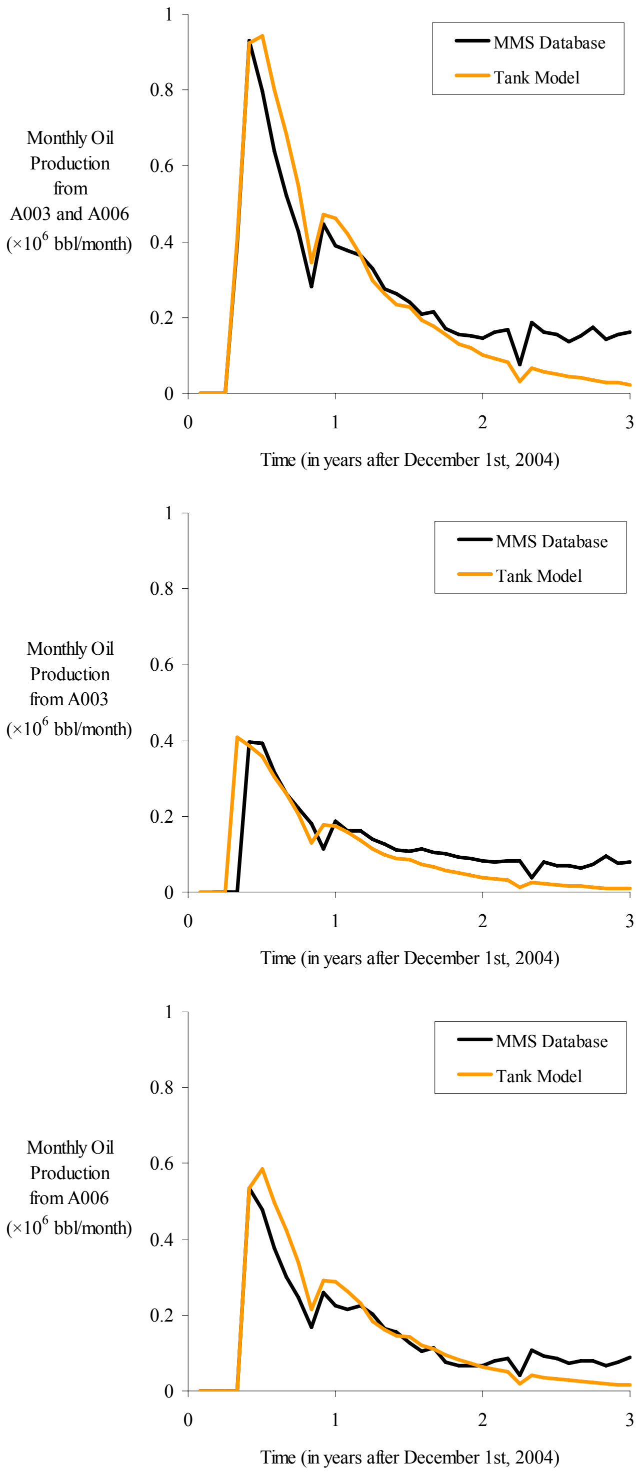
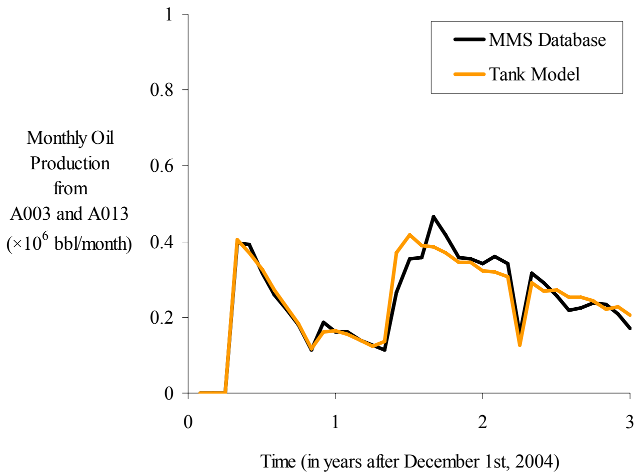


Figure A.6 History matching results: A003 and A006

A003

&

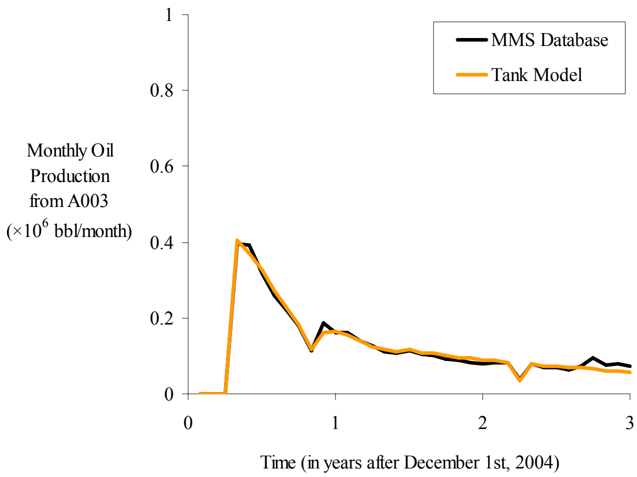
A013



ΔP_{ini}=3375 (psi)

T=3.15 (bbl/psi/day)

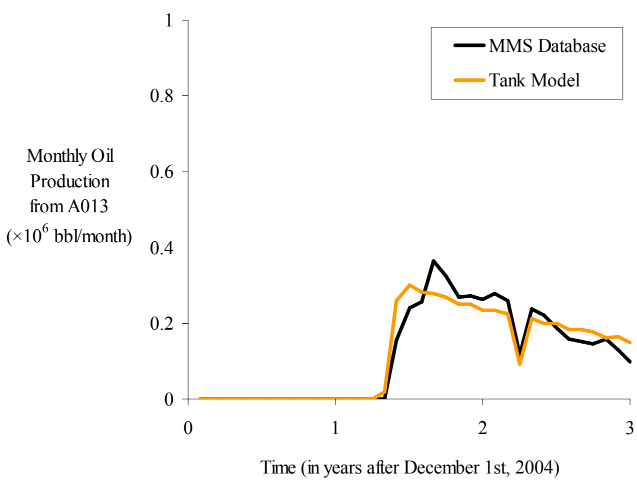
A003



J=11.4 (bbl/psi/day)

A=28.9 (acres)

A013



J=7.9 (bbl/psi/day)

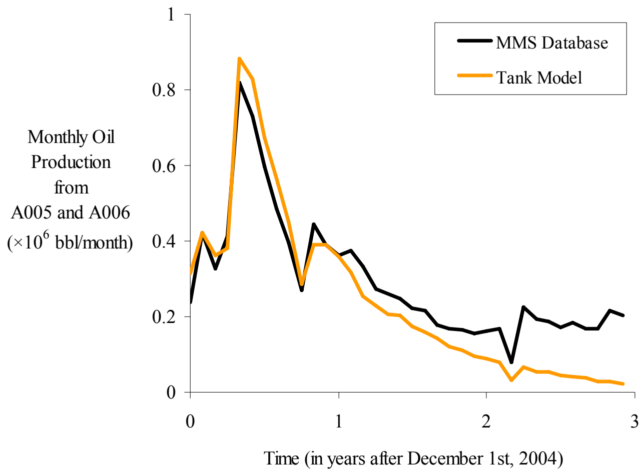
A=97.1 (acres)

Figure A.7 History matching results: A003 and A013

A005

&

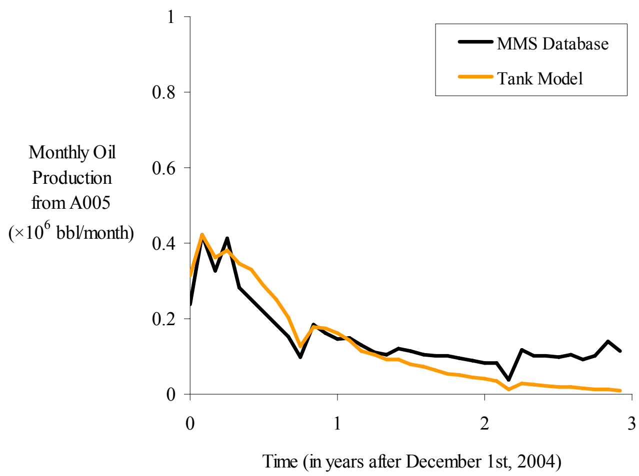
A006



ΔP_{ini}=2572 (psi)

T=59.4 (bbl/psi/day)

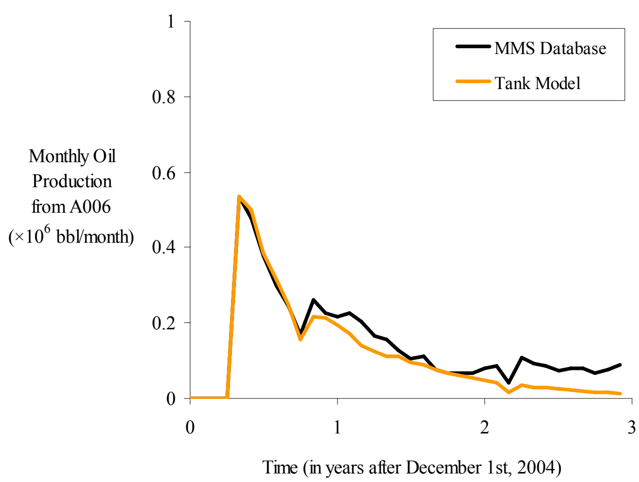
A005



J=25.5 (bbl/psi/day)

A=183 (acres)

A006



J=49.0 (bbl/psi/day)

A=40.9 (acres)

Figure A.8 History matching results: A005 and A006

A005 & A013

ΔP_{ini}=2809 (psi)

T=100 (bbl/psi/day)

A005

J=18.4 (bbl/psi/day)

A=84.3 (acres)

A013

J=25.2 (bbl/psi/day)

A=100 (acres)

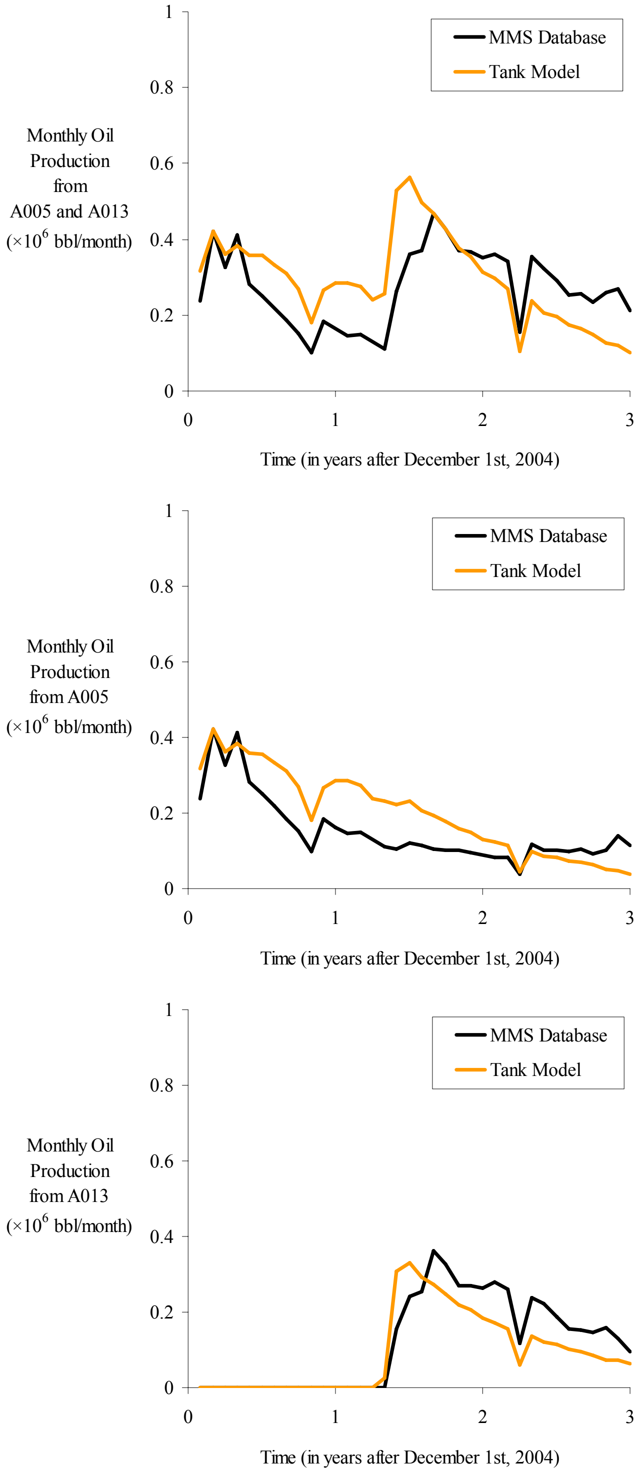
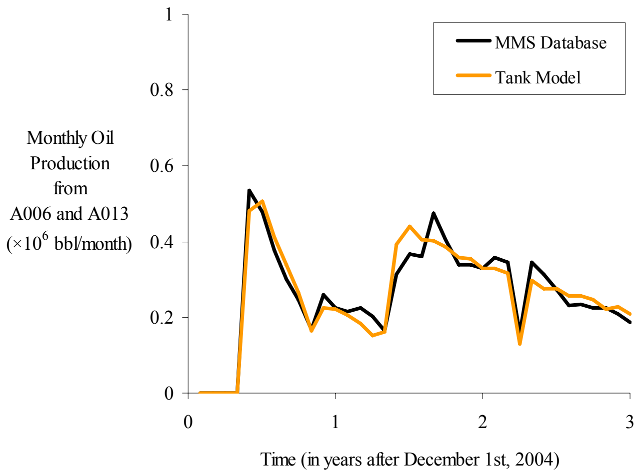


Figure A.9 History matching results: A005 and A013

A006

&

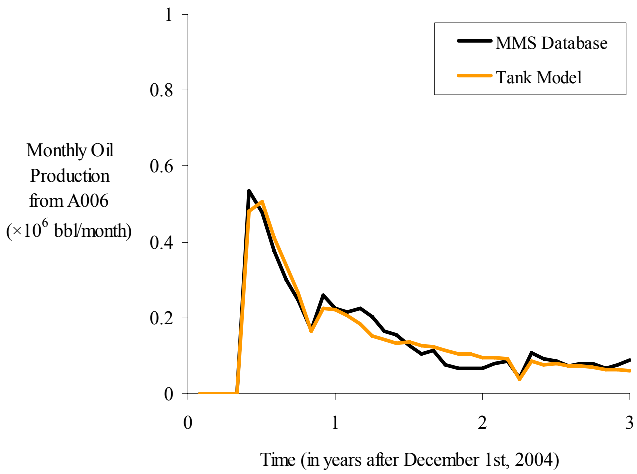
A013



ΔP_{ini}=3338 (psi)

T=3.2 (bbl/psi/day)

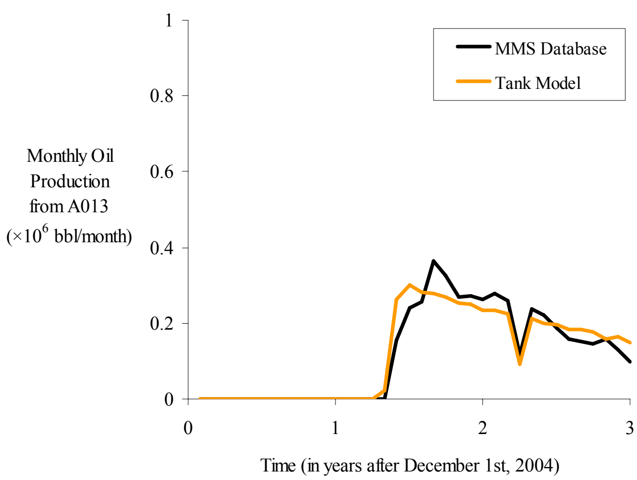
A006



J=15.5 (bbl/psi/day)

A=37.9 (acres)

A013

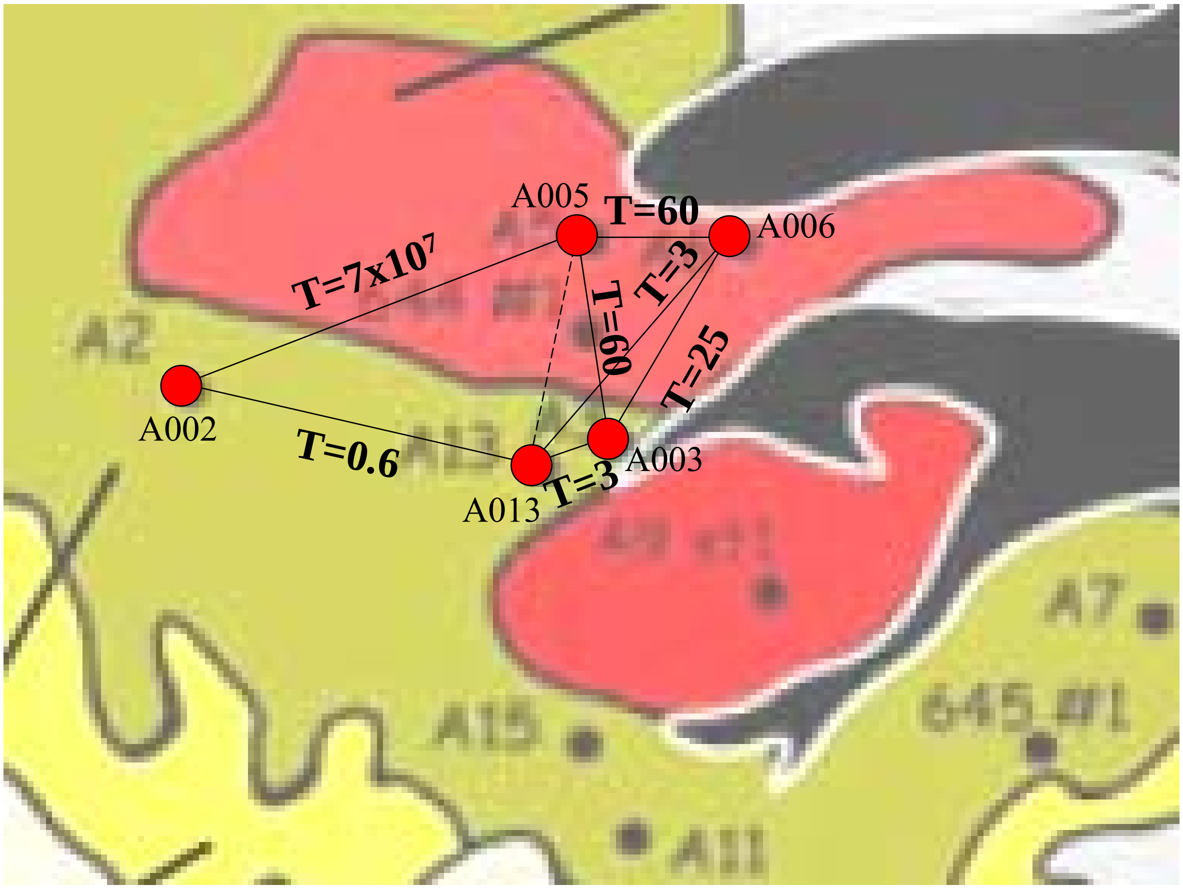


J=8.1 (bbl/psi/day)

A=99.2 (acres)

Figure A.10 History matching results: A006 and A013

The history matching results illustrate the difficulties in obtaining the structure of porous media. The first evidence is a large variation in production histories of the wells in the same geologic structure. The reservoir depositional map in Figure A.1 indicates that the wells, A002, A003, A005, A006, A007, A011, A013, and A015 are in the same channel. However, well performance varies significantly. These wells can be analyzed in groups, one with wells A002, A003, A005, and A006, and the other consisting of wells, A007, A011, and A015. The former group has a higher production rate than the latter group, as shown in Figure A.2. The second evidence is the variation of back-calculated transmissibility values. The back-calculated transmissibilities between each pair of production wells might not correlate with pre-determined geologic structure, as shown in Figure A.11. For example, A002 and A005 are located in two different geologic structures, but history matching yields an extremely large value of transmissibility between the two wells. The large transmissibility implies that the geologic discontinuity between the two wells may not exist. Another example is that A002 and A013 are considered to be at the same channel, but the small transmissibility between them indicates low connectivity. These two evidences of a significant difference in well performance and geologic interpretation illustrate the difficulties in interpreting the heterogeneous porous media.



Transmissibility, T in bbl/psi/day ----------- Partly matched

Figure A.11 Back-calculated transmissibility based on real performance

History matching helped to verify the possible overestimation of physical parameters, such as permeability and net pay of the reservoir. Suppose the wells in Group 2 with small production rates behave ideally. In this ideal condition, each of the production wells, A007, A009, A011, and A015, would have the same production history as A002, which has one of the largest oil recoveries through Holstein's production life so far. This assumption is reasonable because a decision maker who designs the well location may locate production wells as the oil production from each well is not interfered one another. This concept of no interference is equivalent to the isolated wells.

Oil production histories for the idealized Holstein are shown in Figures A.12 and A.13. Considering the design capacity of Holstein, 110,000 bbl/day (3,300,000 bbl/month) (BP America, 2008), the result based on the assumption still gives a much lower production rate. This means that the match between expected and real behavior is not completed only by making low production wells idealized. If the sum of the maximum performance of individual wells does not work, the alternative is to increase maximum performance by increasing physical parameters such as permeability and net pay. This illustrates the difficulties in characterizing heterogeneous porous media.

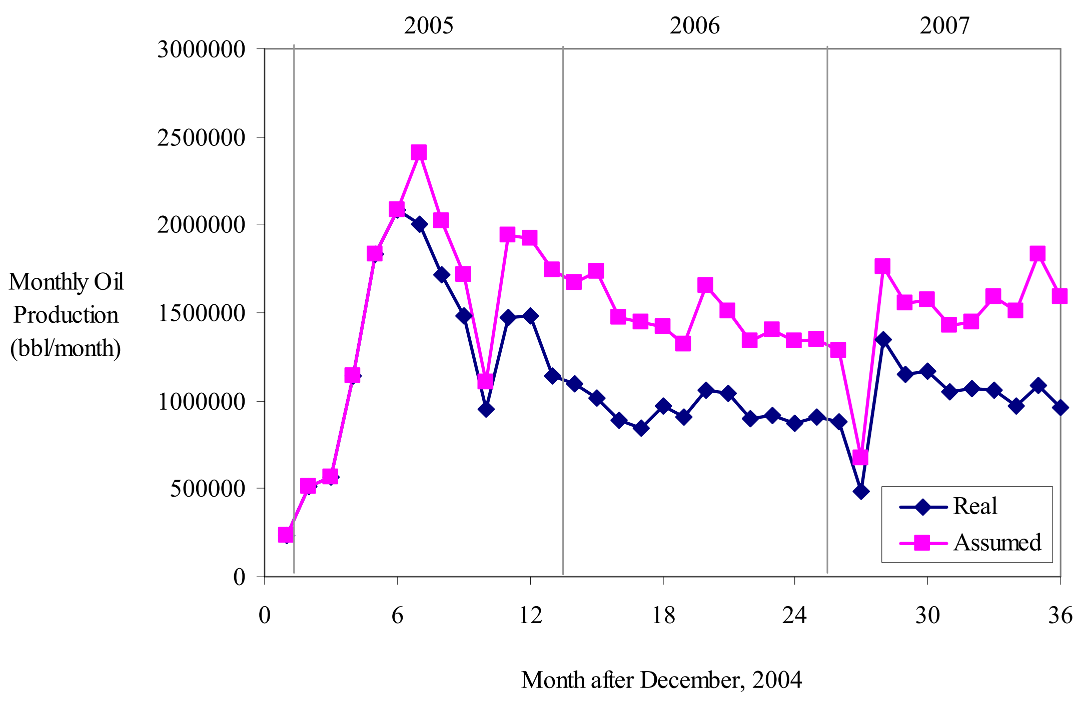


Figure A.12 Monthly oil production for the idealized Holstein oil field

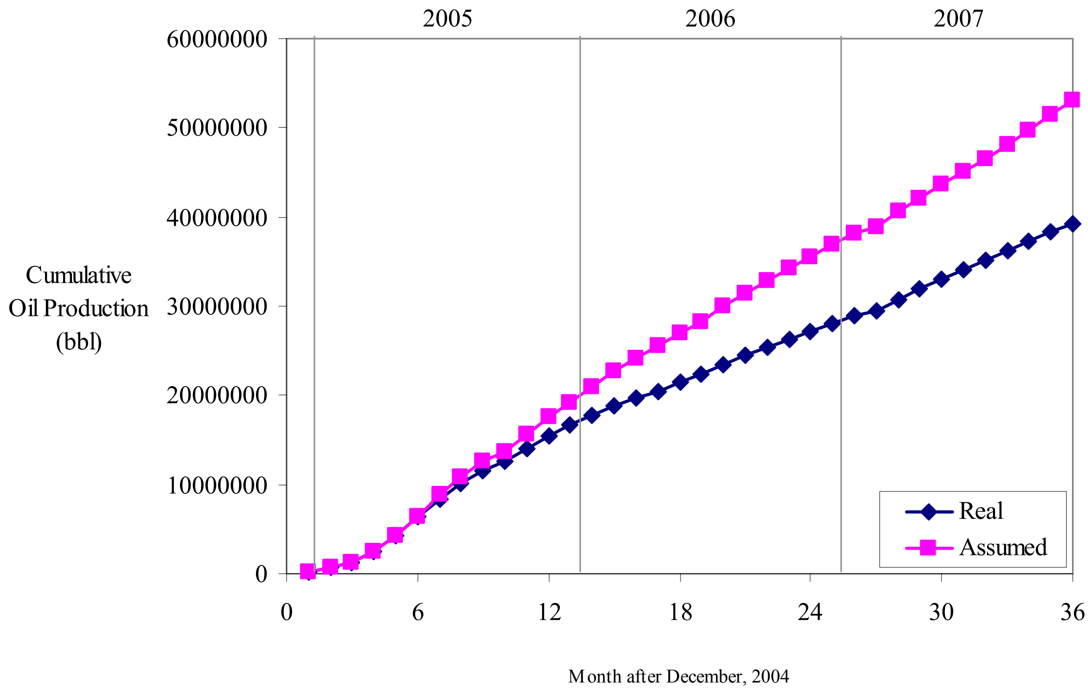


Figure A.13 Cumulative oil production of the idealized Holstein oil field

APPENDIX B. CODE OF THE ALGORITHM FOR DECISION-BASED NON-INFORMATIVE PRIOR PROBABILITIES

The subroutine, noninfo_pmf(), is based on Microsoft Excel Visual Basic Application (VBA) and works for discrete probability distributions. It requires the number of decision alternatives (num_alt), the number of states of nature (num_states), and a decision matrix (decision_matrix) as inputs for the algorithm. The decision matrix is assumed to be located in an Excel sheet named "decision_matrix". The product of this algorithm, non-informative prior probabilities, will be given in the sheet named "noninfoPMF". This code also requires the two sheets named "temp_sorting" and "temp_sorting2" for the purpose of sorting decision alternatives.

The calculation time of the Excel VBA algorithm is shown in Figure B.1. The calculation was performed with a personal computer with 2.4GHz CPU and 3.25 Gb RAM. It is assumed that the numbers of the decision alternatives and the states of nature are equal to each other. The consequences in a decision matrix were generated by a random number generator in Excel. For example, it took 30 minutes for obtaining a decision-based non-informative prior for a decision matrix with 2000 decision alternatives and 2000 states of nature (4,000,000 entries in the decision matrix).

The calculation time for practical decision making would be less than the time took for a decision matrix with random numbers when the numbers of decision alternatives and states of nature are the same for both decision making problems. The reason for this estimation is that a practical decision making problem may have a structure in decision outcomes such as the most preferred decision alternatives and the decision consequences. The difference may reduce the number of preference outcomes, and accordingly, the time for comparing process (this algorithm is based on a simple one-to-one comparison when finding the identical decision outcomes).

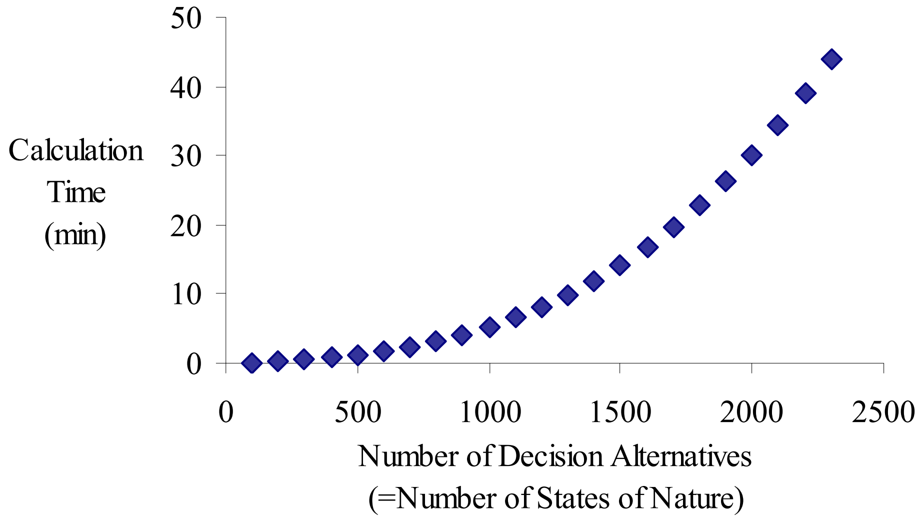


Figure B.1 Calculation time of the algorithm

```
Sub noninfo_pmf()

'Objectives: Assigning noninformative PMF
'Input: num_alt , num_states, Decision matrix in the Excel sheet, "Decision_matrix"
'Output: Decision-based non-informative prior probabilities in the Excel sheet, "noninfoPMF"
'Last update: July 19, 2008
'by Namhong Min

num_alt = 2000
num_states = 2000

ReDim decision_matrix(num_alt, num_states)

'##############################################################################################
'1 Read decision matrix
'##############################################################################################
For i = 1 To num_states
    For j = 1 To num_alt
        decision_matrix(j, i) = Sheets("Decision_matrix").Cells(2 + j, 2 + i)
    Next j
Next i
'##############################################################################################
'2 Lump duplicate columns
'##############################################################################################
ReDim temp_lump(num_alt, num_states)
ReDim decision_matrix_dup_index(num_states)
ReDim temp_decision_matrix_dup_cnt(num_states)

For j = 1 To num_alt
    temp_lump(j, 1) = decision_matrix(j, 1)
```

```
Next j
cnt_lumped = 1
decision_matrix_dup_index(1) = 1
temp_decision_matrix_dup_cnt(1) = 1

For i = 2 To num_states
    cnt_within_tol = 0
    k = 0
    switch_match = 0
    Do While k < cnt_lumped And switch_match = 0
        k = k + 1
        switch_tol1 = 1
        cnt_alt = 1

        Do While switch_tol1 = 1 And cnt_alt <= num_alt
            If Abs(decision_matrix(cnt_alt, i) - temp_lump(cnt_alt, k)) > 0 Then
                switch_tol1 = 0
            Else
                cnt_alt = cnt_alt + 1
            End If
        Loop
        If cnt_alt = (num_alt + 1) And switch_tol1 = 1 Then
            switch_match = 1
            decision_matrix_dup_index(i) = k
            temp_decision_matrix_dup_cnt(k) = temp_decision_matrix_dup_cnt(k) + 1
        End If
    Loop
    If k = cnt_lumped And switch_tol1 = 0 Then
        cnt_lumped = cnt_lumped + 1
        For j = 1 To num_alt
            temp_lump(j, cnt_lumped) = decision_matrix(j, i)
        Next j
        decision_matrix_dup_index(i) = cnt_lumped
        temp_decision_matrix_dup_cnt(cnt_lumped) = 1
    End If
Next i

num_lumped = cnt_lumped

ReDim decision_matrix_dup(num_alt, num_lumped)
ReDim decision_matrix_dup_cnt(num_lumped)
For i = 1 To num_lumped
        decision_matrix_dup_cnt(i) = temp_decision_matrix_dup_cnt(i)
    For j = 1 To num_alt
        decision_matrix_dup(j, i) = temp_lump(j, i)
    Next j
Next i

'####################################################################################################
'3 Build decision outcome matrix: Store the most preferred decision alternative(s) & its consequence
'####################################################################################################
ReDim decision_outcome(num_alt, num_lumped)

Sheets("temp_sorting").Select
    Cells.Select
    Selection.ClearContents
cnt_equal = 0
switch_equal_pref = 0
ReDim equal_pref_lumped(num_lumped) 'storage for the # of lumped scenario which has an equal preference
```

```
For i = 1 To num_lumped
    switch_pref_outcome = 0
    For j = 1 To num_alt
        Sheets("temp_sorting").Cells(j, 1) = j
        Sheets("temp_sorting").Cells(j, 2) = decision_matrix_dup(j, i)
    Next j
    Columns("A:B").Select
    Selection.Sort Key1:=Range("B1"), Order1:=xlDescending, Header:=xlGuess, _
        OrderCustom:=1, MatchCase:=False, Orientation:=xlTopToBottom, _
        DataOption1:=xlSortNormal
    Calculate
    decision_outcome(0, i) = Sheets("temp_sorting").Cells(1, 2)
    j = 1
    Do While switch_pref_outcome = 0 And j < num_alt
        decision_outcome(j, i) = Sheets("temp_sorting").Cells(j, 1)
        If (Sheets("temp_sorting").Cells(j, 2) - Sheets("temp_sorting").Cells(j + 1, 2)) > 0 Then
            decision_outcome(j + 1, i) = 0
            switch_pref_outcome = 1
        ElseIf (Sheets("temp_sorting").Cells(j, 2) - Sheets("temp_sorting").Cells(j + 1, 2)) <= 0 Then
            j = j + 1
        End If
    Loop

    If switch_pref_outcome <> 1 Then
        decision_outcome(num_alt, i) = Sheets("temp_sorting").Cells(num_alt, 1)
    End If
    If j > 1 Then
        For hhh = 1 To j
            Sheets("temp_sorting2").Cells(hhh, 1) = decision_outcome(hhh, i)
        Next hhh
        Sheets("temp_sorting2").Select
        Columns("A:A").Select
        Selection.Sort Key1:=Range("A1"), Order1:=xlAscending, Header:=xlGuess, _
            OrderCustom:=1, MatchCase:=False, Orientation:=xlTopToBottom, _
            DataOption1:=xlSortNormal
        Calculate
        For hhh = 1 To j
            decision_outcome(hhh, i) = Sheets("temp_sorting2").Cells(hhh, 1)
        Next hhh

    End If
    Sheets("temp_sorting").Select
Next i

'###########################################################################################################
'4 Lump duplicated column in decision outcome matrix
'###########################################################################################################
ReDim decision_outcome_dup_index(num_lumped)
ReDim temp_decision_outcome_dup(num_alt, num_lumped)
ReDim temp_decision_outcome_dup_cnt(num_lumped)

temp_decision_outcome_dup(0, 1) = 0 '1
For j = 0 To num_alt
    temp_decision_outcome_dup(j, 1) = decision_outcome(j, 1)
Next j
decision_outcome_dup_index(1) = 1
temp_decision_outcome_dup_cnt(1) = 1
num_decision_outcomes = 1
num_preference_outcomes = 1
For i = 2 To num_lumped
```

```
        switch_new = 1
        switch_new_preference = 1
        For k = 1 To num_decision_outcomes
            switch_iden = 0
            switch_iden_preference = 0
            For j = 0 To num_alt
                If decision_outcome(j, i) = temp_decision_outcome_dup(j, k) Then
                    switch_iden = switch_iden + 1
                    If j <> 0 Then
                        switch_iden_preference = switch_iden_preference + 1
                    End If
                End If
            Next j
            If switch_iden = num_alt + 1 Then
                temp_decision_outcome_dup_cnt(k) = temp_decision_outcome_dup_cnt(k) + 1
                decision_outcome_dup_index(i) = k
                switch_new = 0
            End If

            If switch_iden_preference = num_alt Then
                switch_new_preference = 0
            End If
        Next k
        If switch_new = 1 Then
            num_decision_outcomes = num_decision_outcomes + 1
            temp_decision_outcome_dup_cnt(num_decision_outcomes) = 1
            For j = 0 To num_alt
                temp_decision_outcome_dup(j, num_decision_outcomes) = decision_outcome(j, i)
            Next j
            decision_outcome_dup_index(i) = num_decision_outcomes
        End If
        If switch_new_preference = 1 Then
            num_preference_outcomes = num_preference_outcomes + 1
        End If
    Next i

    ReDim decision_outcome_dup_cnt(num_decision_outcomes)
    ReDim decision_outcome_dup(num_alt, num_decision_outcomes)
    For i = 1 To num_decision_outcomes
        decision_outcome_dup_cnt(i) = temp_decision_outcome_dup_cnt(i)
        For j = 0 To num_alt
            decision_outcome_dup(j, i) = temp_decision_outcome_dup(j, i)
        Next j
    Next i

    '####################################################################################################
    '5 Build preference outcomes
    '####################################################################################################
    ReDim preference_outcome_index(num_decision_outcomes)
    ReDim preference_outcome(num_alt, num_preference_outcomes)
    ReDim preference_outcome_cnt(num_preference_outcomes)
    preference_outcome(0, 1) = 0 '1
    For j = 1 To num_alt
        preference_outcome(j, 1) = decision_outcome_dup(j, 1)
    Next j
    preference_outcome_index(1) = 1
    preference_outcome_cnt(1) = 1
    num_preference_outcomes = 1

    For i = 2 To num_decision_outcomes
```

```
        switch_new = 1
        For k = 1 To num_preference_outcomes
            switch_iden = 0
            For j = 1 To num_alt
                If decision_outcome_dup(j, i) = preference_outcome(j, k) Then
                    switch_iden = switch_iden + 1
                End If
            Next j
            If switch_iden = num_alt Then
                preference_outcome_cnt(k) = preference_outcome_cnt(k) + 1
                preference_outcome_index(i) = k
                switch_new = 0
            End If
        Next k
        If switch_new = 1 Then
            num_preference_outcomes = num_preference_outcomes + 1
            preference_outcome_cnt(num_preference_outcomes) = 1
            For j = 1 To num_alt
                preference_outcome(j, num_preference_outcomes) = decision_outcome_dup(j, i)
            Next j
            preference_outcome_index(i) = num_preference_outcomes
        End If
    Next i

'###############################################################################################
'6 Assigning the noninformative prior probability distribution on lumped scenarios
'###############################################################################################
ReDim pmf_denominator(num_states)
For i = 1 To num_states
    pmf_denominator(i) = num_preference_outcomes
    pmf_denominator(i) = pmf_denominator(i) * decision_matrix_dup_cnt(decision_matrix_dup_index(i))
    pmf_denominator(i) = pmf_denominator(i)_
        _decision_outcome_dup_cnt(decision_outcome_dup_index(decision_matrix_dup_index(i)))
    pmf_denominator(i) = pmf_denominator(i)_
        _*preference_outcome_cnt(preference_outcome_index(decision_outcome_dup_index(decision_matrix_dup_index(i))))
Next i

ReDim pmf(num_states)
temp_pmf_sum = 0
For i = 1 To num_states
    pmf(i) = 1 / pmf_denominator(i)
    temp_pmf_sum = temp_pmf_sum + pmf(i)
Next i
'Print
    Sheets("noninfoPMF").Select
    Columns("A:F").Select
    Selection.ClearContents
Sheets("noninfoPMF").Cells(1, 1) = "States of Nature"
Sheets("noninfoPMF").Cells(1, 2) = "Denominator"
Sheets("noninfoPMF").Cells(1, 3) = "Non-informative prior probabilities"
For i = 1 To num_states
    Sheets("noninfoPMF").Cells(i + 1, 1) = "S" & i
    Sheets("noninfoPMF").Cells(i + 1, 2) = pmf_denominator(i)
    Sheets("noninfoPMF").Cells(i + 1, 3) = pmf(i)
Next i
    Sheets("noninfoPMF").Cells(1, 5) = "Error in PMF_SUM="
    Sheets("noninfoPMF").Cells(1, 6) = (1 - temp_pmf_sum)
End Sub
```

APPENDIX C. MATHEMATICAL DERIVATION OF MODIFIED TANK MODEL

The single-phase fluid flow from two units, as shown in Figure C.1, is modeled with a tank-type model. The model includes a possible connectivity between two units (tanks) with a cross flow rate, q_{XF}.

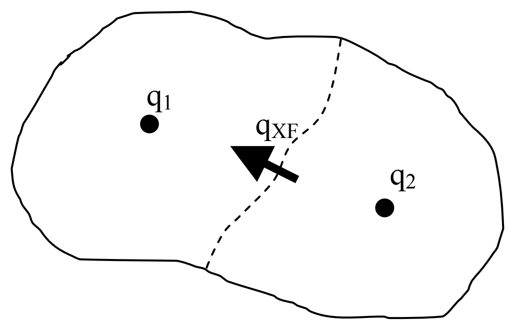


Figure C.1 Flow domain for a tank-type model

The basic equations for modeling the flow include a continuity equation and Darcy's law. These equations, shown in Equations C.1 through C.4, are applied to each unit.

$$V_{P1}c_{t1}\frac{d\overline{P}_1}{dt} = -(q_1 - q_{XF}) \tag{C.1}$$

$$V_{P2}c_{t2}\frac{d\overline{P}_2}{dt} = -(q_2 + q_{XF}) \tag{C.2}$$

$$q_1 = J_1(\overline{P}_1 - P_{wf1}) \tag{C.3}$$

$$q_2 = J_2(\overline{P}_2 - P_{wf2}) \tag{C.4}$$

where V_p is the pore volume, c_t is the total compressibility, $\overline{P}$ is the average reservoir pressure, t is time, q is flow rate, J is the productivity index, and P_{wf} is the wellbore pressure.

These four equations can be expressed with the following three equations.

$$V_{P1}c_{t1}\frac{d\overline{P}_1}{dt} = -J_1\left(\overline{P}_1 - P_{wf1}\right) + q_{XF} \tag{C.5}$$

$$V_{P2}c_{t2}\frac{d\overline{P}_2}{dt} = -J_2\left(\overline{P}_2 - P_{wf2}\right) - q_{XF} \tag{C.6}$$

$$q_{XF} = T\left(\overline{P}_2 - \overline{P}_1\right) \tag{C.7}$$

where T is transmissibility, which quantifies the degree of connectivity between two units. Guillot (1999) developed an analytical solution these equations (Case 1). In this study, the extension of his model is developed in Case 2.

Case 1: No limit on total production rate, q_1+q_2

If there is no limit on the peak production rate, wellbore pressures, P_{wf1} and P_{wf2} are constant with time. By substituting Equations C.7 into Equations C.5 and C.6, the equations can be expressed in a matrix form.

$$\begin{bmatrix} \dfrac{d\overline{P}_1}{dt} \\ \dfrac{d\overline{P}_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{V_{P1}c_{t1}}\left(J_1 + T\right) & \dfrac{1}{V_{P1}c_{t1}}T \\ \dfrac{1}{V_{P2}c_{t2}}T & -\dfrac{1}{V_{P2}c_{t2}}\left(J_2 + T\right) \end{bmatrix} \begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix} + \begin{bmatrix} \dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}} \\ \dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}} \end{bmatrix} \tag{C.8}$$

Here, we end up with a system of nonhomogeneous first-order ordinary differential equations (ODE) with an initial condition, which can be generalized:

$$\hat{X}' = \hat{A}\hat{X} + \hat{f} \tag{C.9}$$

where $\hat{X} = \begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix}$

$$\hat{A} = \begin{bmatrix} -\frac{1}{V_{P1}c_{t1}}(J_1 + T) & \frac{1}{V_{P1}c_{t1}}T \\ \frac{1}{V_{P2}c_{t2}}T & -\frac{1}{V_{P2}c_{t2}}(J_2 + T) \end{bmatrix}$$

$$\hat{f} = \begin{bmatrix} \frac{J_1 P_{wf1}}{V_{P1}c_{t1}} \\ \frac{J_2 P_{wf2}}{V_{P2}c_{t2}} \end{bmatrix}$$

and an initial condition that is $\begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix}_{t=0} = \begin{bmatrix} \overline{P}_1(0) \\ \overline{P}_2(0) \end{bmatrix}$

For flexibility in application of the resultant equations, the initial reservoir pressures for both compartments assumed to be unequal.

The solution of this ODE begins with obtaining eigenvalues. From the determinant in Equation C.10, two eigenvalues and eigenvectors can be calculated, as shown in Equations C.11 and C.12.

$$\begin{aligned} &\det\left(\hat{A} - \lambda I\right) \\ &= \lambda^2 + \left\{\frac{(J_1 + T)}{V_{P1}c_{t1}} + \frac{(J_2 + T)}{V_{P2}c_{t2}}\right\}\lambda + \frac{(J_1 + T)}{V_{P1}c_{t1}}\frac{(J_2 + T)}{V_{P2}c_{t2}} - \frac{T}{V_{P1}c_{t1}}\frac{T}{V_{P2}c_{t2}} \\ &= \lambda^2 + \left\{\frac{(J_1 + T)}{V_{P1}c_{t1}} + \frac{(J_2 + T)}{V_{P2}c_{t2}}\right\}\lambda + \frac{J_1 J_2 + T(J_1 + J_2)}{V_{P1}c_{t1}V_{P2}c_{t2}} \end{aligned} \tag{C.10}$$

$$\lambda_{1,2} = \frac{1}{2}\left[\begin{array}{l} -\left\{\frac{(J_1 + T)}{V_{P1}c_{t1}} + \frac{(J_2 + T)}{V_{P2}c_{t2}}\right\} \\ \pm\sqrt{\left\{\frac{(J_1 + T)}{V_{P1}c_{t1}} - \frac{(J_2 + T)}{V_{P2}c_{t2}}\right\}^2 + 4\left\{\frac{T^2}{V_{P1}c_{t1}V_{P2}c_{t2}}\right\}} \end{array}\right] \tag{C.11}$$

$$\begin{bmatrix} 1 \\ \frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T} \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ \frac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T} \end{bmatrix} \tag{C.12}$$

The term in square root in Equation C.11 is always greater than or equal to zero because it consists of two terms that are greater that or equal to zero. If there is no connectivity between two units (T=0) and $J_1/V_{p1}/c_{t1}$ is equal to $J_2/V_{p2}/c_{t2}$, the eigenvalues are identical to each other. In this case, this eigenvalue approach does not work because the matrix, $\hat{A}$, is already a diagonal matrix. The solution for the case is simply two separated tank models. In the other cases with the term in square root in Equation C.11 greater than zero, there exists two different eigenvalues and following approach is valid. With the eigenvalues, a diagonal matrix, $\hat{\Lambda}$ can be built. The $\hat{P}$ matrix consists of the eigenvectors as in Equation C.14.

$$\hat{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{C.13}$$

$$\hat{P} = \begin{bmatrix} 1 & 1 \\ \frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T} & \frac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T} \end{bmatrix} \tag{C.14}$$

Multiplying the inverse matrix of $\hat{P}$ on both sides in Equation C.9,

$$\hat{P}^{-1}\hat{X}' = \hat{P}^{-1}\hat{A}\,\hat{X} + \hat{P}^{-1}\hat{f} \tag{C.15}$$

Let $\hat{X} = \hat{P}\hat{Y}$ $(\hat{X}' = \hat{P}\hat{Y}')$.

$$\hat{P}^{-1}\hat{P}\hat{Y}' = \hat{P}^{-1}\hat{A}\hat{P}\hat{Y} + \hat{P}^{-1}\hat{f} \tag{C.16}$$

$$\hat{Y}' = \hat{\Lambda}\hat{Y} + \hat{P}^{-1}\hat{f} \tag{C.17}$$

$$\begin{aligned}
\hat{P}^{-1}\hat{f} &= \frac{T}{V_{p1}c_{t1}(\lambda_2 - \lambda_1)} \begin{bmatrix} \dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T} & -1 \\ -\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}} \\ \dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}} \end{bmatrix} \\
&= \frac{T}{V_{p1}c_{t1}(\lambda_2 - \lambda_1)} \begin{bmatrix} \left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(\dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) - \left(\dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right) \\ -\left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(\dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) + \left(\dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right) \end{bmatrix}
\end{aligned} \tag{C.18}$$

Equation C.17 is converted into two simultaneous ODEs, as shown in Equation C.19.

$$\begin{aligned}
\begin{bmatrix} \dfrac{dy_1}{dt} \\ \dfrac{dy_2}{dt} \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&\quad + \frac{T}{V_{p1}c_{t1}(\lambda_2 - \lambda_1)} \begin{bmatrix} \left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(\dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) - \left(\dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right) \\ -\left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(\dfrac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) + \left(\dfrac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right) \end{bmatrix}
\end{aligned} \tag{C.19}$$

For convenience, the constant terms are symbolized as A and B.

$$\frac{dy_1}{dt} = \lambda_1 y_1 + A \tag{C.20}$$

$$\frac{dy_2}{dt} = \lambda_2 y_2 + B \tag{C.21}$$

$$A = \frac{T}{V_{p1}c_{t1}(\lambda_2 - \lambda_1)}\left\{\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(\frac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) - \left(\frac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right)\right\} \tag{C.22}$$

$$B = \frac{T}{V_{p1}c_{t1}(\lambda_2 - \lambda_1)}\left\{-\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(\frac{J_1 P_{wf1}}{V_{P1}c_{t1}}\right) + \left(\frac{J_2 P_{wf2}}{V_{P2}c_{t2}}\right)\right\} \tag{C.23}$$

Solutions for these equations are

$$y_1 = y_{1,\text{Homogeneous}} + y_{1,\text{Particular}} \tag{C.24}$$

$$y_{1,\text{Homogeneous}} = C_1 e^{\lambda_1 t} \tag{C.25}$$

$$y_{1,\text{Particular}} = C_3 \; (\, y'_{1,\text{Particular}} = 0 \,)$$

$$\frac{dy_1}{dt} = \lambda_1 \, y_1 + A \;\; \rightarrow \;\; 0 = \lambda_1 C_3 + A \rightarrow C_3 = -\frac{A}{\lambda_1} \tag{C.26}$$

$$y_1 = C_1 e^{\lambda_1 t} - \frac{A}{\lambda_1} \tag{C.27}$$

$$y_2 = y_{2,\text{Homogeneous}} + y_{2,\text{Particular}} \tag{C.28}$$

$$y_{2,\text{Homogeneous}} = C_2 e^{\lambda_2 t} \tag{C.29}$$

$$y_{2,\text{Particular}} = C_4 \; (\, y'_{2,\text{Particular}} = 0 \,)$$

$$\frac{dy_2}{dt} = \lambda_2 \, y_2 + B \;\; \rightarrow \;\; 0 = \lambda_2 C_4 + B \rightarrow C_4 = -\frac{B}{\lambda_2} \tag{C.30}$$

$$y_2 = C_2 e^{\lambda_2 t} - \frac{B}{\lambda_2} \tag{C.31}$$

Because $\hat{X} = \hat{P}\hat{Y}$,

$$\begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T} & \dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{C.32}$$

$$= \begin{bmatrix} \dfrac{1}{\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}} & \dfrac{1}{\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}} \end{bmatrix} \begin{bmatrix} C_1 e^{\lambda_1 t} - \dfrac{A}{\lambda_1} \\ C_2 e^{\lambda_2 t} - \dfrac{B}{\lambda_2} \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \dfrac{A}{\lambda_1} - \dfrac{B}{\lambda_2} \\ \left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(C_1 e^{\lambda_1 t} - \dfrac{A}{\lambda_1}\right) + \left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(C_2 e^{\lambda_2 t} - \dfrac{B}{\lambda_2}\right) \end{bmatrix}$$

From the initial conditions on average reservoir pressure,

$$C_1 = \frac{\left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\overline{P}_1(0) - \overline{P}_2(0)}{\left(\dfrac{V_{p1}c_{t1}\lambda_2 - V_{p1}c_{t1}\lambda_1}{T}\right)} + \frac{A}{\lambda_1} \tag{C.33}$$

$$C_2 = \frac{\left(\dfrac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\overline{P}_1(0) - \overline{P}_2(0)}{\left(\dfrac{V_{p1}c_{t1}\lambda_1 - V_{p1}c_{t1}\lambda_2}{T}\right)} + \left(\frac{B}{\lambda_2}\right) \tag{C.34}$$

From the solution with the same initial average reservoir pressure in both units, cross flow rate, production rate, and oil recovery for each unit can be calculated.

$$\begin{aligned} q_{XF} &= T\left(\overline{P}_2 - \overline{P}_1\right) \\ &= \left(J_1 + V_{p1}c_{t1}\lambda_1\right)\left(C_1 e^{\lambda_1 t} - \frac{A}{\lambda_1}\right) + \left(J_1 + V_{p1}c_{t1}\lambda_2\right)\left(C_2 e^{\lambda_2 t} - \frac{B}{\lambda_2}\right) \end{aligned} \tag{C.35}$$

$$q_1 = J_1\left(\overline{P}_1 - P_{wf1}\right) = J_1\left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \frac{A}{\lambda_1} - \frac{B}{\lambda_2} - P_{wf1}\right) \tag{C.36}$$

$$q_2 = J_2\left(\overline{P}_2 - P_{wf2}\right)$$

$$= J_2\left\{\begin{aligned}&\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(C_1 e^{\lambda_1 t} - \frac{A}{\lambda_1}\right)\\ &\quad+\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(C_2 e^{\lambda_2 t} - \frac{B}{\lambda_2}\right) - P_{wf2}\end{aligned}\right\} \qquad \text{(C.37)}$$

Oil recovery from the time, t_1 to the time t_2, can be calculated by integrating the equations for the production rate, which is given by the Equations, C.36 and C.37. The oil recovery from each unit, R_1 and R_2, is given in following equations.

$$R_1 = J_1\left\{\begin{aligned}&\frac{C_1 e^{\lambda_1 t_2} - C_1 e^{\lambda_1 t_1}}{\lambda_1} + \frac{C_2 e^{\lambda_2 t_2} - C_2 e^{\lambda_2 t_1}}{\lambda_2}\\ &-\left(\frac{A}{\lambda_1} + \frac{B}{\lambda_2} + P_{wf1}\right)\left(t_2 - t_1\right)\end{aligned}\right\} \qquad \text{(C.38)}$$

$$R_2 = J_2\left\{\begin{aligned}&\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_1}{T}\right)\left(\frac{C_1 e^{\lambda_1 t_2} - C_1 e^{\lambda_1 t_1}}{\lambda_1} - \frac{A}{\lambda_1}\left(t_2 - t_1\right)\right)\\ &+\left(\frac{J_1 + T + V_{p1}c_{t1}\lambda_2}{T}\right)\left(\frac{C_2 e^{\lambda_2 t_2} - C_2 e^{\lambda_2 t_1}}{\lambda_2} - \frac{B}{\lambda_2}\left(t_2 - t_1\right)\right)\\ &- P_{wf2}\left(t_2 - t_1\right)\end{aligned}\right\} \qquad \text{(C.39)}$$

Case 2 Limit on total production rate, q_1+q_2

When flow rate is limited, the wellbore pressure is also a function of time. Therefore, the basic equations, Equations C.5, C.6, and C.7, must be solved differently.

$$q_{Lim} = q_1(t) + q_2(t) = J_1(\overline{P}_1 - P_{wf1}) + J_2(\overline{P}_2 - P_{wf2}) \tag{C.40}$$

$$J_1 P_{wf1} + J_2 P_{wf2} = J_1 \overline{P}_1 + J_2 \overline{P}_2 - q_{Lim} \tag{C.41}$$

It is assumed that borehole pressure difference remains the same ($P_{wf1}=P_{wf2}+\Delta P_{wf}$).

$$J_1 P_{wf1} + J_2 (P_{wf1} - \Delta P_{wf}) = J_1 \overline{P}_1 + J_2 \overline{P}_2 - q_{Lim} \tag{C.42}$$

$$J_1 (P_{wf2} + \Delta P_{wf}) + J_2 P_{wf2} = J_1 \overline{P}_1 + J_2 \overline{P}_2 - q_{Lim} \tag{C.43}$$

$$P_{wf1} = \frac{J_1}{J_1 + J_2} \overline{P}_1 + \frac{J_2}{J_1 + J_2} \overline{P}_2 + \frac{(J_2 \Delta P_{wf} - q_{Lim})}{J_1 + J_2} \tag{C.44}$$

$$P_{wf2} = \frac{J_1}{J_1 + J_2} \overline{P}_1 + \frac{J_2}{J_1 + J_2} \overline{P}_2 + \frac{(-J_1 \Delta P_{wf} - q_{Lim})}{J_1 + J_2} \tag{C.45}$$

The governing equations can be expressed as follows.

$$\begin{bmatrix} \dfrac{d\overline{P}_1}{dt} \\ \dfrac{d\overline{P}_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{V_{P1} c_{t1}} \left(\dfrac{J_1 J_2}{J_1 + J_2} + T \right) & \dfrac{1}{V_{P1} c_{t1}} \left(\dfrac{J_1 J_2}{J_1 + J_2} + T \right) \\ \dfrac{1}{V_{P2} c_{t2}} \left(\dfrac{J_1 J_2}{J_1 + J_2} + T \right) & -\dfrac{1}{V_{P2} c_{t2}} \left(\dfrac{J_1 J_2}{J_1 + J_2} + T \right) \end{bmatrix} \begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{V_{P1} c_{t1}} \dfrac{J_1 (J_2 \Delta P_{wf} - q_{Lim})}{J_1 + J_2} \\ \dfrac{1}{V_{P2} c_{t2}} \dfrac{J_2 (-J_1 \Delta P_{wf} - q_{Lim})}{J_1 + J_2} \end{bmatrix} \tag{C.46}$$

The initial conditions are identical to Case 1. The solution for Case 2 can be derived in the same way as Case 1. However, Case 2 always has two different eigenvalues because the determinant in Equation C.46 is equal to zero only if all J_1, J_2 and T are equal to zero, which is physically meaningless. The two eigenvalues are shown in Equations C.47 through C.49.

$$\det(\overline{A}-\lambda\bar{I})=\lambda^2+\left\{\frac{1}{V_{P1}c_{t1}}\left(\frac{J_1J_2}{J_1+J_2}+T\right)+\frac{1}{V_{P2}c_{t2}}\left(\frac{J_1J_2}{J_1+J_2}+T\right)\right\}\lambda=0 \tag{C.47}$$

$$\lambda\left[\lambda+\left\{\frac{1}{V_{P1}c_{t1}}\left(\frac{J_1J_2}{J_1+J_2}+T\right)+\frac{1}{V_{P2}c_{t2}}\left(\frac{J_1J_2}{J_1+J_2}+T\right)\right\}\right]=0 \tag{C.48}$$

$$\lambda_1=0$$

$$\lambda_2=-\frac{1}{V_{P1}c_{t1}}\left(\frac{J_1J_2}{J_1+J_2}+T\right)-\frac{1}{V_{P2}c_{t2}}\left(\frac{J_1J_2}{J_1+J_2}+T\right) \tag{C.49}$$

$$\begin{aligned}\overline{P}_1=&-\frac{\left(\dfrac{q_{Lim}}{J_1+J_2}\right)\left(\dfrac{J_1+J_2}{V_{P2}c_{t2}}\right)}{\left(\dfrac{V_{P1}c_{t1}}{V_{P2}c_{t2}}+1\right)}t+C_1\\&+C_2\exp\left[-\left(\frac{J_1J_2}{J_1+J_2}+T\right)\left(\frac{1}{V_{P1}c_{t1}}+\frac{1}{V_{P2}c_{t2}}\right)t\right]\\&+\frac{\dfrac{1}{V_{P1}c_{t1}}\dfrac{\left(J_1J_2\Delta P_{wf}-J_1q_{Lim}\right)}{J_1+J_2}+\dfrac{1}{V_{P2}c_{t2}}\dfrac{\left(J_1J_2\Delta P_{wf}+J_2q_{Lim}\right)}{J_1+J_2}}{\left(\dfrac{V_{P1}c_{t1}}{V_{P2}c_{t2}}+1\right)\left(\dfrac{J_1J_2}{J_1+J_2}+T\right)\left(\dfrac{1}{V_{P1}c_{t1}}+\dfrac{1}{V_{P2}c_{t2}}\right)}\end{aligned} \tag{C.50}$$

$$\begin{aligned}\overline{P}_2=&-\frac{\left(\dfrac{q_{Lim}}{J_1+J_2}\right)\left(\dfrac{J_1+J_2}{V_{P2}c_{t2}}\right)}{\left(\dfrac{V_{P1}c_{t1}}{V_{P2}c_{t2}}+1\right)}t+C_1-\frac{V_{P1}c_{t1}}{V_{P2}c_{t2}}\times\\&\left\{\begin{aligned}&C_2\exp\left[-\left(\frac{J_1J_2}{J_1+J_2}+T\right)\left(\frac{1}{V_{P1}c_{t1}}+\frac{1}{V_{P2}c_{t2}}\right)t\right]\\&+\frac{\dfrac{1}{V_{P1}c_{t1}}\dfrac{\left(J_1J_2\Delta P_{wf}-J_1q_{Lim}\right)}{J_1+J_2}+\dfrac{1}{V_{P2}c_{t2}}\dfrac{\left(J_1J_2\Delta P_{wf}+J_2q_{Lim}\right)}{J_1+J_2}}{\left(\dfrac{V_{P1}c_{t1}}{V_{P2}c_{t2}}+1\right)\left(\dfrac{J_1J_2}{J_1+J_2}+T\right)\left(\dfrac{1}{V_{P1}c_{t1}}+\dfrac{1}{V_{P2}c_{t2}}\right)}\end{aligned}\right\}\end{aligned} \tag{C.51}$$

$$C_1 = \frac{\frac{V_{P1}c_{t1}}{V_{P2}c_{t2}}\overline{P}_1(0) + \overline{P}_2(0)}{\left(1 + \frac{V_{P1}c_{t1}}{V_{P2}c_{t2}}\right)} \tag{C.52}$$

$$C_2 = \frac{\overline{P}_1(0) - \overline{P}_2(0)}{\left(\frac{V_{P1}c_{t1}}{V_{P2}c_{t2}} + 1\right)} - \frac{\frac{1}{V_{P1}c_{t1}}\frac{\left(J_1 J_2 \Delta P_{wf} - J_1 q_{Lim}\right)}{J_1 + J_2} + \frac{1}{V_{P2}c_{t2}}\frac{\left(J_1 J_2 \Delta P_{wf} + J_2 q_{Lim}\right)}{J_1 + J_2}}{\left(\frac{V_{P1}c_{t1}}{V_{P2}c_{t2}} + 1\right)\left(\frac{J_1 J_2}{J_1 + J_2} + T\right)\left(\frac{1}{V_{P1}c_{t1}} + \frac{1}{V_{P2}c_{t2}}\right)} \tag{C.53}$$

APPENDIX D. CODE OF THE FLOW SIMULATOR USED FOR THE SPATIAL VARIABILITY EXAMPLE

The code below was used to simulate a one phase flow in 4 by 4 grid flow domain, shown in Section 7.4.2. The calculation is straightforward because the code is implicit and makes use of only matrix inversion and multiplication. Results of the simulator include histories of reservoir pressure (pressure at each grid) and wellbore pressure, and total recovery from production well(s). For convenience in economic analysis possibly following the reservoir simulation, the discount factor is multiplied to oil recovery.

```
clear all;
poro=0.3;
ct=0.00000000725;                      % (1/Pa)
mu=0.0008;                             % (Pa sec)
skin=0;                                %(-)
pini=(2200)*(1000/0.1450);             % initial pressure in N/m^2, ()in psi
pwf=(2000)*(1000/0.1450);              % wellbore pressure in N/m^2, ()in psi
delt=1*60*60*24*5;                     % time increment in sec
end_t=1*60*60*24*365*20;               % total production period
delz=(300)*0.3048;                     % net pay, height of cubics in m, () in ft
grid_area=(200)*4046.8/(16);           %Area for one grid in m^2, (A_total) in acres, (N)#of grids
vb=grid_area*(delz);                   %bulk volume of ONE GRID in m^3,    (h) in ft
ca=30.88;                              %at the center of a rectangular grid
qlim=(1000)*1/6.289/24/60/60;          %production rate limit, m^3/sec () in STB/Day
disc_rate=0.05;                        % discount rate
k(1)=(2)*(9.8692*0.0000000000000001);  % m^2, 1 (md)=9.8692*10^-16 (m^2)
k(2)=(2)*(9.8692*0.0000000000000001);
k(3)=(2)*(9.8692*0.0000000000000001);
k(4)=(2)*(9.8692*0.0000000000000001);
k(5)=(2)*(9.8692*0.0000000000000001);
k(6)=(2)*(9.8692*0.0000000000000001);
k(7)=(2)*(9.8692*0.0000000000000001);
k(8)=(2)*(9.8692*0.0000000000000001);
k(9)=(2)*(9.8692*0.0000000000000001);
k(10)=(2)*(9.8692*0.0000000000000001);
k(11)=(2)*(9.8692*0.0000000000000001);
k(12)=(2)*(9.8692*0.0000000000000001);
k(13)=(2)*(9.8692*0.0000000000000001);
k(14)=(2)*(9.8692*0.0000000000000001);
k(15)=(2)*(9.8692*0.0000000000000001);
k(16)=(2)*(9.8692*0.0000000000000001);
well_loc=6;                            % the grid where production wells are located
num_well=1;                            % the number of wells

temp_oil_rcvry=zeros(0,1);
recovery=zeros(3,3);
```

```
bbb=zeros(16,16);
p_hist=pini*ones(16,1);
Trans=zeros(16,16);
inv_trans=zeros(16,16);
Trans(1,1)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(1)+1/k(2))+1/(1/k(1)+1/k(5)));
Trans(1,2)=-2*delt*delz/mu*(1/(1/k(1)+1/k(2)));
Trans(1,5)=-2*delt*delz/mu*(1/(1/k(1)+1/k(5)));
Trans(2,1)=Trans(1,2);
Trans(2,2)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(2)+1/k(1))+1/(1/k(2)+1/k(3))+1/(1/k(2)+1/k(6)));
Trans(2,3)=-2*delt*delz/mu*(1/(1/k(2)+1/k(3)));
Trans(2,6)=-2*delt*delz/mu*(1/(1/k(2)+1/k(6)));
Trans(3,2)=Trans(2,3);
Trans(3,3)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(3)+1/k(2))+1/(1/k(3)+1/k(4))+1/(1/k(3)+1/k(7)));
Trans(3,4)=-2*delt*delz/mu*(1/(1/k(3)+1/k(4)));
Trans(3,7)=-2*delt*delz/mu*(1/(1/k(3)+1/k(7)));
Trans(4,3)=Trans(3,4);
Trans(4,4)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(4)+1/k(3))+1/(1/k(4)+1/k(8)));
Trans(4,8)=-2*delt*delz/mu*(1/(1/k(4)+1/k(8)));
Trans(5,1)=Trans(1,5);
Trans(5,5)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(5)+1/k(1))+1/(1/k(5)+1/k(6))+1/(1/k(5)+1/k(9)));
Trans(5,6)=-2*delt*delz/mu*(1/(1/k(5)+1/k(6)));
Trans(5,9)=-2*delt*delz/mu*(1/(1/k(5)+1/k(9)));
Trans(6,2)=Trans(2,6);
Trans(6,5)=Trans(5,6);
Trans(6,6)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(6)+1/k(2))+1/(1/k(6)+1/k(5))+1/(1/k(6)+1/k(7))+1/(1/k(6)+1/k(10)));
Trans(6,7)=-2*delt*delz/mu*(1/(1/k(6)+1/k(7)));
Trans(6,10)=-2*delt*delz/mu*(1/(1/k(6)+1/k(10)));
Trans(7,3)=Trans(3,7);
Trans(7,6)=Trans(6,7);
Trans(7,7)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(7)+1/k(3))+1/(1/k(7)+1/k(6))+1/(1/k(7)+1/k(8))+1/(1/k(7)+1/k(11)));
Trans(7,8)=-2*delt*delz/mu*(1/(1/k(7)+1/k(8)));
Trans(7,11)=-2*delt*delz/mu*(1/(1/k(7)+1/k(11)));
Trans(8,4)=Trans(4,8);
Trans(8,7)=Trans(7,8);
Trans(8,8)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(8)+1/k(4))+1/(1/k(8)+1/k(7))+1/(1/k(8)+1/k(12)));
Trans(8,12)=-2*delt*delz/mu*(1/(1/k(8)+1/k(12)));
Trans(9,5)=Trans(5,9);
Trans(9,9)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(9)+1/k(5))+1/(1/k(9)+1/k(10))+1/(1/k(9)+1/k(13)));
Trans(9,10)=-2*delt*delz/mu*(1/(1/k(9)+1/k(10)));
Trans(9,13)=-2*delt*delz/mu*(1/(1/k(9)+1/k(13)));
Trans(10,6)=Trans(6,10);
Trans(10,9)=Trans(9,10);
Trans(10,10)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(10)+1/k(6))+1/(1/k(10)+1/k(9))+1/(1/k(10)+1/k(11))+1/(1/k(10)+1/k(14)));
Trans(10,11)=-2*delt*delz/mu*(1/(1/k(10)+1/k(11)));
Trans(10,14)=-2*delt*delz/mu*(1/(1/k(10)+1/k(14)));
Trans(11,7)=Trans(7,11);
Trans(11,10)=Trans(10,11);
Trans(11,11)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(11)+1/k(7))+1/(1/k(11)+1/k(10))+1/(1/k(11)+1/k(12))+1/(1/k(11)+1/k(15)));
Trans(11,12)=-2*delt*delz/mu*(1/(1/k(11)+1/k(12)));
Trans(11,15)=-2*delt*delz/mu*(1/(1/k(11)+1/k(15)));
Trans(12,8)=Trans(8,12);
Trans(12,11)=Trans(11,12);
Trans(12,12)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(12)+1/k(8))+1/(1/k(12)+1/k(11))+1/(1/k(12)+1/k(16)));
Trans(12,16)=-2*delt*delz/mu*(1/(1/k(12)+1/k(16)));
Trans(13,9)=Trans(9,13);
Trans(13,13)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(13)+1/k(9))+1/(1/k(13)+1/k(14)));
Trans(13,14)=-2*delt*delz/mu*(1/(1/k(13)+1/k(14)));
Trans(14,10)=Trans(10,14);
Trans(14,13)=Trans(13,14);
Trans(14,14)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(14)+1/k(10))+1/(1/k(14)+1/k(13))+1/(1/k(14)+1/k(15)));
```

```
Trans(14,15)=-2*delt*delz/mu*(1/(1/k(14)+1/k(15)));
Trans(15,11)=Trans(11,15);
Trans(15,14)=Trans(14,15);
Trans(15,15)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(15)+1/k(11))+1/(1/k(15)+1/k(14))+1/(1/k(15)+1/k(16)));
Trans(15,16)=-2*delt*delz/mu*(1/(1/k(15)+1/k(16)));
Trans(16,12)=Trans(12,16);
Trans(16,15)=Trans(15,16);
Trans(16,16)=poro*vb*ct+2*delt*delz/mu*(1/(1/k(16)+1/k(12))+1/(1/k(16)+1/k(15)));
prod_ind=num_well*2*3.1415926*k(well_loc)*delz/mu/(0.5*log(4*grid_area/num_well/ca/0.1524/0.1524/(exp(0.5772)))+skin);

% q>qlim? Try pressure specific first
p_old=p_hist(:,1)*poro*vb*ct;
Trans(well_loc,well_loc)=Trans(well_loc,well_loc)+delt*prod_ind;
p_old(well_loc)=p_old(well_loc)+delt*prod_ind*pwf;
inv_trans=inv(Trans);
p_trial=inv_trans * p_old;
q_trial=prod_ind*(p_trial(well_loc)-pwf);
disc_oil_rcvry=0;
time_step=0;
time_step_qlim=0;

% If qini is estimated to be larger than qlim, use a constant-q model
if q_trial>qlim
    Trans(well_loc,well_loc)=Trans(well_loc,well_loc)-delt*prod_ind;
    inv_trans=inv(Trans);
end
while (q_trial>qlim) && (time_step*delt<end_t)
    time_step=time_step+1;
    p_old=p_hist(:,time_step)*poro*vb*ct;
    p_old(well_loc)=p_old(well_loc)-delt*qlim;
    p_hist=horzcat(p_hist,inv_trans * p_old);
    disc_oil_rcvry=disc_oil_rcvry+qlim*delt*(1/(1+disc_rate))^(1+fix(delt*time_step/60/60/24/30/12-1/60/60/24/30/12));
    temp_oil_rcvry=vertcat(temp_oil_rcvry,qlim*delt);
    q_trial=prod_ind*(p_hist(well_loc,time_step+1)-pwf);
    oil_rcvry_qlim=disc_oil_rcvry;
    time_step_qlim=time_step;
end

%Use constant-p model
if time_step>1 && (time_step*delt<end_t)
    disc_oil_rcvry=disc_oil_rcvry-qlim*delt*(1/(1+disc_rate))^(1+fix(delt*time_step/60/60/24/30/12-1/60/60/24/30/12));
    p_hist=p_hist(:,1:time_step);
    Trans(well_loc,well_loc)=Trans(well_loc,well_loc)+delt*prod_ind;
    p_old(well_loc)=p_old(well_loc)+delt*prod_ind*pwf+delt*qlim;
    inv_trans=inv(Trans);
    q_trial=qlim*0.9; % 0.9 is an arbitrary number to make q_trial less than qlim for next production phase
    time_step=time_step-1;
end
while (q_trial<qlim) && (time_step*delt<end_t)
    time_step=time_step+1;
    time_step_decay=time_step-time_step_qlim;
    p_old=p_hist(:,time_step)*poro*vb*ct;
    p_old(well_loc)=p_old(well_loc)+delt*prod_ind*pwf;
    p_hist=horzcat(p_hist,inv_trans * p_old);
    disc_oil_rcvry=disc_oil_rcvry+delt*prod_ind*(0.5*(p_hist(well_loc,time_step)+p_hist(well_loc,time_step+1))-
            pwf)*(1/(1+disc_rate))^(1+fix(delt*time_step/60/60/24/30/12-1/60/60/24/30/12));
  temp_oil_rcvry=vertcat(temp_oil_rcvry,delt*prod_ind*(0.5*(p_hist(well_loc,time_step)+p_hist(well_loc,time_step+1))-pwf));
    q_trial=prod_ind*(p_hist(well_loc,time_step+1)-pwf);
end
recovery(1,num_well)=disc_oil_rcvry;
```

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